A NOTE ON QUADRICS OVER A FINITE FIELD

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1. Let F denote the finite field of odd order q. Eckford Cohen [1] has proved the following results.

I. Let S_n denote an n-dimensional affine space with F as base field. If $n \geq 4$ there are no hyperplanes of S_n contained in the complement of the quadric $Q_n(a)$ defined by

$$a_1x_1^2 + \cdots + a_nx_n^2 = a \qquad (a_1 \cdots a_n \neq 0).$$

II. Let T_n denote an n-dimensional projective space with base field F. If $n \geq 3$, a quadric Q_n of T_n defined by

$$a_0x_0^2 + a_1x_1^2 + \cdots + a_nx_n^2 = 0 \qquad (a_0a_1 \cdots a_n \neq 0)$$

has at least one point in common with a given hyperplane

$$b_0x_0 + b_1x_1 + \cdots + b_nx_n = 0.$$

Let Q_n denote the quadric of T_n defined by

$$a_0x_0^2 + a_1x_1^2 + \cdots + a_nx_n^2 = 0 \qquad (a_0a_1 \cdots a_n \neq 0). \tag{1.1}$$

There is no loss in generality in assuming that the quadratic form in (1.1) is in diagonal form. If ψ (a) denotes the nonprincipal quadratic character of F, that is $\psi(a) = +1, -1$ or 0 according as a is a square, a nonsquare or zero in F, then we define the *exterior* of Q_n as the set of points $(x_0, x_1, \cdots, x_n)$ of T_n such that

$$\psi(Q(x_0, x_1, \cdots, x_n)) = +1.$$

Similarly the *interior* of Q_n is the set of points of T_n such that

$$\psi(Q(x_0, x_1, \cdots, x_n)) = -1.$$

For a given hyperplane L_n defined by

$$b_0x_0 + b_1x_1 + \cdots + b_nx_n = 0 \tag{1.2}$$

we let $N_E(L_n)$ denote the number of points of L_n in the exterior of Q_n and $N_I(L_n)$ the number of points of L_n in the interior of Q_n. The numbers $N_E(L_n)$ and $N_I(L_n)$ are determined explicitly below (see Theorem 1). Moreover we find as a corollary of the theorem that $N_E(L_n) = N_I(L_n)$ or $N_E(L_n) + N_I(L_n) = q^{n-1}$. Finally (Theorem 4) we determine the number of points in the interior and in the exterior of Q_n.

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