A VARIATIONAL METHOD FOR UNIVALENT QUASICONFORMAL MAPPINGS

By M. Schiffer

The purpose of this paper is the derivation of a formal calculus of variations which deals with extremum problems in the theory of quasi-conformal mappings and which is a natural extension of an analogous general method in the theory of conformal mappings [7], [8], [9], [10]. While the property of quasi-conformality allows one only to assert the Hölder continuity of the mapping function, the fact that it solves certain extremum problems leads often to the conclusion that it is closely related to an analytic function. P. P. Belinski [2], [3], [4] has developed a variational procedure of a similar kind and made numerous applications of it. Considering properly chosen extremum problems, he could give existence proofs for interesting quasi-conformal mappings, show their analytic character and derive useful estimates for the general class considered.

The calculus derived here has the merit of leading to a simple algorithm which is the natural extension of an analogous theory in the case of conformal mapping. We unite in a single procedure the treatment of extremum problems in both cases. The method leads in a very elementary way to necessary extremum conditions for a smooth quasiconformal competing function. Much more interesting is the fact that the same method leads to an existence proof for such smooth extremal quasiconformal functions.

The method is exemplified in one explicit extremum problem which shows the power of the variational technique.

1. We consider univalent mappings $w = f(z, \bar{z}) = u(x, y) + iv(x, y)$ of a domain D_z in the z-plane onto the domain D_w in the w-plane which are of the class C^1. We call the mapping K-quasiconformal in D_z if it satisfies the inequality

$$|f_z|^2 + |f_{\bar{z}}|^2 \leq K(|f_z|^2 - |f_{\bar{z}}|^2) \qquad K > 1 \tag{1}$$

for all points of D_z. We observe the identities

$$|f_z|^2 + |f_{\bar{z}}|^2 = \tfrac{1}{2}(u_x^2 + u_y^2 + v_x^2 + v_y^2) \tag{2}$$

and

$$|f_z|^2 - |f_{\bar{z}}|^2 = u_x v_y - u_y v_x = \frac{\partial(u, v)}{\partial(x, y)} = J \tag{3}$$

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