SUBORDINATE H^p FUNCTIONS

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Dedicated to Professor Charles Loewner

1. Although it was Littlewood who introduced the terminology, one may go back at least to Lindelöf's work [7] in 1908 and find application of the subordination principle. The fact that the range of one function is contained in that of a second does not furnish much information relating the two functions in general. However, if we restrict our attention to certain regular (= analytic) functions which agree at a single point, very sharp comparisons can be made. We mention here only the works of Littlewood [8], Rogosinski [14], [15], Schiffer [17] and Golusin [3] who exploited the idea with marked success.

Suppose then that F is regular and *univalent* in the disc $D = \{|z| < 1\}$ and let f be a second function, regular in D, whose range is contained in that of F, with $f(0) = F(0)$. We say that f is *subordinate* to F and write $f \prec F$. The function $\phi = F^{-1}(f)$ maps D into itself and is subject to the inequality $|\phi(z)| \leq |z|$, from the Schwarz Lemma. Thus, $f(z) = F(\phi(z))$ for $z \in D$ and suggests that we drop the condition that F be univalent and simply require that $f \prec F$ whenever $f = F(\phi)$ for some regular ϕ which satisfies $|\phi(z)| \leq |z|$ in D. One should take note that this extended definition does not imply that functions with the same range can be related by subordination. The condition is more stringent than that.

Perhaps the Schwarz Lemma itself was the first result concerning subordinate functions. In [8] Littlewood showed that if $f \prec F$ and $p > 0$, then

$$(1) \qquad \mathfrak{M}_p[f; r] = \left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p} \leq \left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta\right\}^{1/p} = \mathfrak{M}_p[F; r]$$

whenever $0 < r < 1$. One may also obtain limiting inequalities by making $p \to 0$ or $p \to \infty$. Moreover, the integrals (1) are increasing functions of r and therefore have limits as $r \to 1$. If $\lim_{r \to 1} \mathfrak{M}_p[F; r] < \infty$, we write this limit as $||F||_p$ (although it does not give a norm when $0 < p < 1$). When $||F||_p$ exists, there exist radial limits $F(e^{i\theta}) = \lim_{r \to 1} F(re^{i\theta})$ for almost all $\theta \in [0, 2\pi)$ (in fact they exist on a set of class $F_{\sigma\delta}$ with measure 2π). The resulting function, which we also write as F, is of class $L^p(0, 2\pi)$ and

$$\lim_{r \to 1} \mathfrak{M}_p[F; r] = \left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{i\theta})|^p \, d\theta\right\}^{1/p}.$$

We denote the class of regular functions in D for which $||F||_p$ exists by H^p. A complete discussion of the Banach spaces H^p for $p \geq 1$ may be found in [4].

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