CONNECTED $G\delta$ GRAPHS

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All functions considered in this paper have domain I (the unit interval) and range contained in I; the word "graph" means the graph of such a function.

It is well known that if a function f is the pointwise limit of a sequence of continuous functions, then the graph of f is a G_δ set in I^2 (the unit square). It is natural to ask whether the converse of this result, or a weakened form of the converse, is true.

Question 1. *Is a function whose graph is a connected G_δ set in I^2 the pointwise limit of a sequence of continuous functions?*

This question leads in a natural way to

Question 2. *If U is an open subset of the plane containing a graph which is a connected G_δ set, then does U contain the graph of a continuous function?*

Although the two questions relate closely to the same problem, they are independent.

The main result of this paper is an example which answers both questions in the negative. In addition, connected G_δ graphs are shown to be dense in certain "nice" continua and it is proved that, in connection with Question 1, one can obtain pointwise convergence on a dense G_δ set in I. A modification of the example shows that, in general, this is "best possible."

Notation. We use standard notation for subintervals of I and for points of I^2; whether (x, y) denotes the open interval from x to y in I or a point of I^2 will be clear from the context. We use the cross product notation where convenient; for example, $[a, b] \times \{c\} = \{(x, y) \in I^2 \mid a \leq x \leq b \text{ and } y = c\}$. Finally, for each x in I, the set $\{x\} \times I$ will be denoted l_x.

1. The closure of a connected G_δ graph. Let G be a connected G_δ graph; we first prove that G is nowhere dense in I^2. Let $[a, b] \times [c, d]$ be a nondegenerate rectangle in I^2 and let $G_1, G_2, \cdots$ be a nested sequence of (relatively) open subsets of I^2 whose intersection is G. For each positive integer n and each pair of rational numbers r, s such that $c \leq r < s \leq d$, let $H(n, r, s)$ denote the set of points x in $[a, b]$ such that some component of $l_x - G_n$ meets both $I \times \{r\}$ and $I \times \{s\}$. Since the G_i are open, each $H(n, r, s)$ is closed in $[a, b]$, and since G is a graph, every point of $[a, b]$ belongs to some $H(n, r, s)$. Thus there exist n, r and s such that $H(n, r, s)$ contains an interval (e, f). Then G misses the

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