THE SIZE OF THE RIEMANN ZETA-FUNCTION AT PLACES SYMMETRIC WITH RESPECT TO THE POINT 1/2

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In a recent paper, Spira [2] proves that for $t \geq 10$ and $\frac{1}{2} < \sigma < 1$, the inequality $|\zeta(1-s)| > |\zeta(s)|$ holds except at the zeros of $\zeta(s)$; as usual, $s = \sigma + it$. Here we give a simpler proof of the following stronger version:

Theorem . *If* $|t| \geq 6.8$ *and* $\sigma > \frac{1}{2}$, *then* $|\zeta(1-s)| > |\zeta(s)|$ *except at the zeros of* $\zeta(s)$.

Proof. The familiar functional equation is $\zeta(1-s) = g(s)\zeta(s)$ where

$$g(s) = 2(2\pi)^{-s}\Gamma(s) \cos \pi s/2 \qquad (=\chi(1-s)).$$

Clearly, $g(s)$ is analytic for $t \neq 0$ so that $|g(s)|$ is continuous for such s; putting $s_0 = \frac{1}{2} + it$ and noting that

$$|g(s_0)| = |\zeta(\tfrac{1}{2} - it)|/|\zeta(\tfrac{1}{2} + it)| = 1$$

holds for each t such that $\zeta(\frac{1}{2} + it) \neq 0$, the continuity of $|g(s)|$ guarantees that $|g(s_0)| = 1$ holds for all real $t \neq 0$. Defining $h(s) = \log |g(s)/g(s_0)|$, it therefore suffices to prove that $h(s) > 0$ whenever $\sigma > \frac{1}{2}$ and $|t| \geq 6.8$ since the inequality for $h(s)$ implies that $|g(s)| > 1$.

From now on, let $\sigma > \frac{1}{2}$ and $t \neq 0$. We have

$$h(s) = \log|\Gamma(s)| - \log|\Gamma(s_0)| + \tfrac{1}{2}\log\frac{\cosh \pi t + \cos \pi\sigma}{\cosh \pi t} - (\sigma - \tfrac{1}{2}) \log 2\pi$$

$$\geq (\sigma - \tfrac{1}{2})\left\{\frac{\partial}{\partial\sigma} \log|\Gamma(\sigma + it)|\right\}_{\sigma=\sigma_1} + \tfrac{1}{2}\log\left\{1 - \frac{|\sin \pi(\sigma - \frac{1}{2})|}{\cosh \pi t}\right\} - (\sigma - \tfrac{1}{2}) \log 2\pi$$

where, by the law of the mean, σ_1 is a suitable number between $\frac{1}{2}$ and σ. Now $\frac{1}{2}\log(1 - x) + x$ has a non-negative derivative for $x \leq \frac{1}{2}$ and is zero at $x = 0$; hence it is non-negative for $0 \leq x \leq \frac{1}{2}$ so that

$$\tfrac{1}{2}\log\left\{1 - \frac{|\sin \pi(\sigma - \frac{1}{2})|}{\cosh \pi t}\right\} \geq -\frac{|\sin \pi(\sigma - \frac{1}{2})|}{\cosh \pi t} > -2\pi\frac{\sigma - \frac{1}{2}}{e^{\pi|t|}}$$

provided $|t| \geq \frac{1}{2}$. Consequently,

$$\frac{h(s)}{\sigma - \frac{1}{2}} > \left\{\frac{\partial}{\partial\sigma}\log|\Gamma(\sigma + it)|\right\}_{\sigma=\sigma_1} - 2\pi e^{-\pi|t|} - \log 2\pi. \tag{1}$$