HYPERBOLIC CAPACITY AND INTERPOLATING RATIONAL FUNCTIONS II

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A close connection between the capacity of a point set in the euclidean plane and extremal polynomials defined on the set was first indicated by Fekete, and later developed in more detail by Leja and Walsh; these polynomials have important applications in interpolation to analytic functions; see for instance [4, Chapter 7]. Tsuji [3] indicated more recently an analogous property of certain extremal rational functions and their connection with the *hyperbolic capacity* of a point set contained in the unit disk (hyperbolic plane), a property studied recently in more detail by Pommerenke [2] and Walsh [6]. The latter made application of these results to interpolation by rational functions. The object of the present note is to establish an analog in the hyperbolic case of later results of Leja [1] concerning extremal polynomials in the euclidean plane, and to make applications also to interpolation. Namely, Leja defined a sequence of points $\alpha_1, \alpha_2, \cdots$, such that $\alpha_1, \alpha_2, \cdots, \alpha_n$ form an extremal set (a set with certain extremal properties) for every n, and these can be used to define a *series of interpolation*

$$a_0 + a_1(z - \alpha_1) + a_2(z - \alpha_1)(z - \alpha_2) + \cdots,$$

an expansion of an arbitrary analytic function rather than merely a sequence of polynomials of interpolation. We establish here the validity in the hyperbolic plane of a corresponding series of interpolation of rational functions (hyperbolic polynomials) for an arbitrary analytic function.

To be more explicit, let Γ denote the unit circumference $|z| = 1$, E with boundary B a closed set interior to Γ, whose complement K with respect to H: $|z| < 1$ is connected and is regular in the sense that the classical Dirichlet problem for K has a solution. Let the points $\alpha_1, \alpha_2, \alpha_3, \cdots$ be defined as follows: α_1 is an arbitrary point of E; α_2 is a point of E for which the hyperbolic distance $[\alpha_2, \alpha_1] = |(\alpha_2 - \alpha_1)/(1 - \bar{\alpha}_1\alpha_2)| = \max |(z - \alpha_1)/(1 - \bar{\alpha}_1 z)|$, z on E; α_3 a point of E for which $[\alpha_3, \alpha_1]\cdot[\alpha_3, \alpha_2] = \max \prod_{k=1}^{2} |(z - \alpha_k)/(1 - \bar{\alpha}_k z)|$, z on E; and in general, α_{n+1} a point of E for which

$$(1) \qquad [\alpha_{n+1}, \alpha_1]\cdot[\alpha_{n+1}, \alpha_2] \cdots [\alpha_{n+1}, \alpha_n] = \max \prod_{k=1}^{n} \left| \frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \right|, \qquad z \text{ on } E.$$

Any such sequence of points is an *extremal* set for E. All points $\alpha_1, \alpha_2, \alpha_3, \cdots$, except possibly α_1, lie on the boundary B of E since $\prod_{k=1}^{n} (z - \alpha_k)/(1 - \bar{\alpha}_k z)$

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