SETS OF VISIBLE POINTS

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Most classic lattice-point problems in the theory of numbers have originated from one or the other of two sources: Minkowski's Theorem, and the Gauss Circle Problem. Minkowski's Theorem may be classified as an existence theorem—from information about the size and shape of a region we deduce the existence of lattice points inside the region. The Gauss Circle Problem is a counting problem. There is no question of existence here. A circle of large radius has many lattice points inside it—the problem is to find out more about the number of these lattice points.

This paper deals with lattice-point problems of an entirely different nature. These might be called *visibility problems.* We say that two points of the n-dimensional integer lattice L^n are mutually visible or can see one another if there is no lattice point on the open line segment joining them. If Q is a subset of L^n, we write VQ for the set of points in L^n which can see every point of Q, and we say that a set S is a set of visible points if $S = VQ$ for some set $Q \leq L^n$.

If x and y are two points of L^n and m is a positive integer, we write $x \equiv y(m)$ to mean that m divides each component of the vector $x - y$. It is easy to see that x and y are mutually visible if and only if the greatest common divisor of the components of $x - y$ is 1 (we stipulate that a point cannot see itself). An immediate consequence is that x and y are mutually visible if and only if $x \not\equiv y(p)$ for all primes p. This suggests the following definition: two points $x, y \in L^n$ are mutually visible modulo the prime p if $x \not\equiv y(p)$. If Q is a subset of L^n, we write V_pQ to indicate the set of all points in L^n which can see every point of Q modulo p. The equation

$$VQ = \bigcap_p V_pQ \tag{1}$$

follows immediately from these definitions.

We shall be interested in finding the "density" of the set VQ (if it has one). However, even without a definition of the "density" of a set, we can make some plausible remarks about the "probability that a point of L^n is in VQ." (We shall not attempt to justify the use of the language of probability theory. However, all the statements made in this paragraph can be formulated precisely in terms of the density of a set and most of them will be proved in our subsequent analysis. For a detailed discussion of this idea see Kac [3]). First consider the case where Q consists of a single point q. It seems reasonable to assign the event "x can see q modulo p" the probability $1 - p^{-n}$, since q is

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