SINGULAR MEASURES AND DOMAINS NOT OF SMIRNOV TYPE

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The existence of rectifiably bounded Jordan domains not of Smirnov type has been proved only through a complicated geometric construction of M. V. Keldyš and M. A. Lavrentiev. We present here, among other things, a characterization of the singular measures associated with such domains. The main theorem, when combined with "real-variables" constructions due to G. Piranian and to J.-P. Kahane, gives new examples much simpler than that of Keldyš and Lavrentiev.

1. The main result. Let D be a domain in the complex plane bounded by a rectifiable Jordan curve, and let $f(z)$ map $|z| < 1$ conformally onto D. The derivative $f'(z)$ is then of class H^1 and has no zeros, so it has a canonical factorization [10; 78] of the form

$$f'(z) = e^{i\gamma} \exp\left\{-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right\} \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log |f'(e^{it})|\, dt\right\},$$

where γ is a real number and $\mu(t)$ is a bounded non-decreasing singular function: $\mu'(t) = 0$ a.e. The exponential expression involving μ will be called the *singular factor*.

The presence of a (non-trivial) singular factor is a property only of the domain D, and not of the particular mapping function f [10; 160]. D is said to be a *Smirnov domain* if there is no singular factor; that is, if $d\mu = 0$. Smirnov domains are the well-behaved ones from the standpoint of approximation theory and polynomial expansion. (See [11], [12], [3; 396].)

The question naturally arises: Is *every* rectifiably bounded Jordan domain a Smirnov domain? In 1937, Kelydš and Lavrentiev [6] gave a negative answer by presenting a counterexample. Their construction was simplified in the book of Privalov [10; 162–181], but the technical details are still quite formidable. The domain D is produced as the union of an expanding sequence of domains generated inductively by a prescribed set of rules. It is shown that the mapping function $f(z)$ has the properties $|f'(z)| \leq 1$ in $|z| < 1$ and $|f'(z)| = 1$ almost everywhere on $|z| = 1$, so that, with the normalization $f'(0) > 0$,

$$f'(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right\}. \tag{1}$$

The construction is geometric, and it gives no information about the singular

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