# APPLICATIONS OF $M$-MATRICES TO NON-NEGATIVE MATRICES

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**1.** A matrix $B = (b_{ij})$ of order $n$ is called non-negative (written $B \geq 0$) if each element of $B$ is a non-negative number. If each element is positive, then $B$ is called a positive matrix (written $B > 0$). Generalizing some theorems of Perron [14] on positive matrices, Frobenius [7], [8], [9] showed that every non-negative matrix $B$ has a non-negative characteristic root $p(B)$ (the *Perron root* of $B$) such that each characteristic root $\beta$ of $B$ satisfies $|\beta| \leq p(B)$.

Since the publication of the results of Perron and Frobenius, the problem of finding estimates for $p(B)$ has been studied extensively. For a history of the problem, see Brauer [1], [2] and Taussky [15]. Some well-known results in this area are as follows (see e.g. [10]). Let $B \geq 0$, and for $i = 1, 2, \cdots, n$ let $R_i(B)$ denote the sum of the elements in row $i$. Let $R(B)$ be the largest, and $r(B)$ the smallest, of the $R_i(B)$. Then

$$r(B) \leq p(B) \leq R(B). \tag{1}$$

Also

$$p(B) \geq b_{ii} \qquad (i = 1, 2, \cdots, n). \tag{2}$$

If each of $B$ and $C$ is a matrix of order $n$, we write $B \geq C$ to mean $(B - C) \geq 0$.

$$\text{If} \quad B \geq C \geq 0, \quad \text{then} \quad p(B) \geq p(C). \tag{3}$$

Finally,

$$\text{if } B' \text{ is a principal submatrix of } B, \text{ then } p(B) \geq p(B'). \tag{4}$$

Closely related to non-negative matrices is a class of matrices called $M$-matrices. A square matrix $A$ is called an $M$-matrix if it has the form $kI - B$, where $B$ is a non-negative matrix, $k > p(B)$, and $I$ denotes the identity matrix. Ostrowski [12], [13] first studied $M$-matrices, and they have since been investigated by Fan [4], [5] and Fiedler and Pták [6]. In case $A$ is a real, square matrix with non-positive off-diagonal elements, each of the following is a necessary and sufficient condition for $A$ to be an $M$-matrix (see e.g. [6]).

$$\text{Each principal minor of } A \text{ is positive.} \tag{5}$$

$$A \text{ is non-singular, and } A^{-1} \geq 0. \tag{6}$$

$$\text{Each real characteristic root of } A \text{ is positive.} \tag{7}$$

Ky Fan [4] proved the following lemma.

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