REGULAR D-CLASSES WHOSE IDEMPOTENTS OBEY CERTAIN CONDITIONS

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To Professor A. D. Wallace on his sixtieth birthday

Let S be a semigroup and A a non-empty subset of S. Let E_A denote the collection of idempotents of A. E_A may be partially ordered as follows: $e \leq f$ if and only if $ef = fe = e$. In a previous paper [6], we characterized regular D-classes D such that E_D was linearly ordered. The purpose of this paper is to consider weaker assumptions than a linear order on E_D . In §1, we characterize regular D-classes D for which E_D is directed from above (e, f in E_D imply there exists g in E_D such that $g \geq e$ and $g \geq f$). This section is due to Professor A. H. Clifford. In §2, we essentially consider the case where E_D is a band whose principal ideals are linearly ordered. A significant generalization of the main result of [6] is given, and the D-classes D such that E_D is a left regular band in the sense of Kimura [5] are characterized. Basic definitions may be found in [3]; likewise references to the fundamental work of Anderson, Clifford, Croisot, Green, Koch, McLean, Miller, Munn, Penrose, Preston, Thierrin, and Wallace may be found in [3].

1. Regular D-classes D for which E_D is directed from above. R, L, H, and D denote Green's relations [3; 47]. S^1 will denote S with appended identity [3; 4].

Lemma 1.1. *A regular D-class D of a semigroup S is a subsemigroup of S if and only if $E_D^2 \subseteq D$. D is then a regular bisimple subsemigroup.*

Proof. The 'only if' is trivial. Assume that $E_D^2 \subseteq D$, and let a and b be any two elements of D. Then, there exist e, f in E_D such that aLe and bRf [3; 58]. Thus, ab in D [3, Theorem 2.4; 49]. Then, D is bisimple [3; 62, Example 6]. D is a regular semigroup [3, Lemma 1.14; 27 and Theorem 2.18; 60].

Lemma 1.2. *Let e be an idempotent element of a semigroup S and let a, b be in eSe. Then a and b are D-equivalent in eSe if and only if they are D-equivalent in S. Moreover if aRc and cLb, then c in eSe, and aRc, cLb relative to eSe.*

Proof. If a and b are D-equivalent relative to eSe, they are obviously also D-equivalent relative to S. Hence, assume that aDb relative to S. Then there exists c in S such that aRc and cLb. We shall show incidentally that any such c belongs to eSe. Now, $ax = c$, $cy = a$, $ub = c$, and $vc = b$ for some x, y, u, and v in S^1. Thus, from $ea = a$, we get $ec = eax = ax = c$, and similarly from

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