THE FOURIER SERIES OF GEGENBAUER'S FUNCTION

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1. Introduction. If N is a positive integer, the Gegenbauer polynomial C_N^ν is known to have the representation [3, vol. 2; 175]

$$C_N^\nu(\cos\theta) = \sum_{m=0}^{N} \frac{(\nu)_m(\nu)_{N-m}}{m!\,(N-m)!} \cos(N-2m)\theta, \tag{1.1}$$

where $(\nu)_m = \Gamma(\nu+m)/\Gamma(\nu)$. The Fourier series of Gegenbauer's function $C_\alpha^\nu(\cos\theta)$ with general (possibly complex) α does not appear to have been given previously, even in the special case of Legendre's function $P_\alpha = C_\alpha^{1/2}$. We shall find that

$$\begin{gathered} C_\alpha^\nu(\cos\theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta, \qquad (\operatorname{Re}\nu < 1), \\ A_n = \left\{1 + \frac{\sin\pi(\nu+\alpha+n)}{\sin\pi\nu}\right\} \frac{\Gamma\left(\nu + \frac{\alpha+n}{2}\right)\Gamma\left(\nu + \frac{\alpha-n}{2}\right)}{[\Gamma(\nu)]^2\Gamma\left(1 + \frac{\alpha+n}{2}\right)\Gamma\left(1 + \frac{\alpha-n}{2}\right)}. \end{gathered} \tag{1.2}$$

If $\operatorname{Re}\nu \geq 1$, the Fourier coefficients do not exist for general α because $C_\alpha^\nu(\cos\theta)$ is not integrable over an interval containing the point $\theta = \pi$. If $\operatorname{Re}\nu < 1$ and α is a positive integer N, the first factor of A_n vanishes if $N+n$ is odd; (1.2) then reduces to (1.1).

We remark that Gegenbauer's function (multiplied by a constant to give it the value unity at $\theta = 0$) has a nicely symmetrical expression in the notation of the hypergeometric R function [1]:

$$\begin{aligned} \frac{\Gamma(2\nu)\Gamma(\alpha+1)}{\Gamma(2\nu+\alpha)} C_\alpha^\nu(\cos\theta) &= R(-\alpha;\nu,\nu;e^{i\theta},e^{-i\theta}) \\ &= 2^{\nu-\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2})(\sin\theta)^{\frac{1}{2}-\nu}P_{\alpha+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cos\theta), \end{aligned} \tag{1.3}$$

where P is an associated Legendre function. An important special case is

$$C_\alpha^{\frac{1}{2}}(\cos\theta) = R(-\alpha;\tfrac{1}{2},\tfrac{1}{2};e^{i\theta},e^{-i\theta}) = P_\alpha(\cos\theta). \tag{1.4}$$

2. The Fourier coefficients. Gegenbauer's function is defined [3, vol. 1;

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