SOME DIFFERENCE EQUATIONS

By L. Carlitz

1. J. A. Morrison [2] has considered the functional-difference equation

$$(1.1)\qquad (x-\alpha)(\alpha-\beta)^{n-1}g_n(x) = \alpha(x-\beta)^n g_{n-1}(\alpha) - x(\alpha-\beta)^n g_{n-1}(x) \qquad (n = 1, 2, 3, \cdots)$$

with $g_0(x) = 1$ and $0 < \alpha < \beta$, and has proved that

$$(1.2)\qquad g_n(\alpha) = \frac{1}{n}\sum_{r=0}^{n-1}\binom{n}{r}\binom{n}{r+1}\alpha^r\beta^{n-r}.$$

The writer [1] has introduced the coefficients $A_r^{(n)}$ occurring in

$$(1.3)\qquad g_n(x) = \sum_{r=0}^{n-1} A_r^{(n)}(\alpha-\beta)^{-r}(x-\beta)^r \qquad (n = 1, 2, 3, \cdots).$$

Riordan [3] has proved the explicit result:

$$(1.4)\qquad A_r^{(n)} = (n-r)\sum_{j=1}^{r}\frac{1}{j}\binom{n-1}{j-1}\binom{r-1}{j-1}\alpha^j\beta^{n-j} \qquad (1 \le r < n);$$

it is known that

$$(1.5)\qquad A_0^{(n)} = \beta^n.$$

Put $A_r^{(n)} = \beta^n a_{nr}$. Riordan obtained (1.4) by solving the difference equation

$$(1.6)\qquad a_{nr} - a_{n-1,r} - a_{n,r-1} + a_{n-1,r-1} = \lambda a_{n-1,r-1},$$

where $0 \le r < n$ and $a_{nn} = \delta_{n0}$; $\lambda = \alpha/\beta$.

In the present note we consider the equation (1.6) for all $n > 1$, $r > 1$ and such that

$$(1.7)\qquad a_{n1} = (n-1)\lambda, \qquad a_{1n} = -(n-1)\lambda.$$

2. Put

$$(2.1)\qquad F(x, y) = \sum_{n=1}^{\infty}\sum_{r=1}^{\infty} a_{nr}x^n y^r.$$

Then, by (1.6) and (1.7), we have

$$(1-x)(1-y)F(x, y) = \lambda xyF(xy) + (1-x)\sum_{n=1}^{\infty} a_{n1}x^n y + (1-y)\sum_{r=1}^{\infty} a_{1r}xy^r,$$

$$[(1-x)(1-y) - \lambda xy]F(x, y) = \frac{\lambda x^2 y}{1-x} - \frac{\lambda xy^2}{1-y} = \frac{\lambda xy(x-y)}{(1-x)(1-y)},$$

Received November 3, 1964. Supported in part by NSF grant GP-1593.