A FUNCTIONAL-DIFFERENCE EQUATION

By John Riordan

1. J. A. Morrison [2] has considered the functional-difference equation

$$(1.1) \qquad (x - \alpha)(\alpha - \beta)^{n-1} g_n(x) = \alpha(x - \beta)^n g_{n-1}(\alpha) - x(\alpha - \beta)^n g_{n-1}(x), \qquad n = 1, 2, \cdots$$

with $g_0(x) = 1$ and $0 < \alpha < \beta$, and has proved that

$$g_n(\alpha) = \beta^n \sum_{r=0}^{n-1} \binom{n}{r}\binom{n-1}{r} \frac{y^r}{r+1}, \qquad y = \alpha/\beta.$$

L. Carlitz [1] has looked for an explicit formula for $g_n(x)$, or more specifically for the coefficients $A_r^{(n)}$ in

$$(1.2) \qquad g_n(x) = \sum_{r=0}^{n-1} A_r^{(n)} w^r, \qquad w = (x - \beta)/(\alpha - \beta).$$

Here it is shown that

$$(1.3) \qquad A_r^{(n)} = \sum_{j=1}^{r} \frac{1}{j} \binom{n-1}{j-1}\binom{r-1}{j-1}(n-r)\beta^n y^j, \qquad r = 1, 2, \cdots, n-1.$$

As Carlitz has shown, $A_0^{(n)} = \beta^n$.

2. It is convenient to write $A_r^{(n)} = \beta^n a_{nr}$. Substituting (1.2) into (1.1) leads to

$$(2.1) \qquad (w - 1)g_n(x) = \alpha w^n g_{n-1}(\alpha) - [\beta + (\alpha - \beta)w]g_{n-1}(x).$$

Then since $g_n(\alpha) = \Sigma A_r^{(n)} = \beta^n \Sigma a_{nr}$, it follows at once that

$$(2.2) \qquad a_{nr} - a_{n,r-1} = a_{n-1,r} + (y - 1)a_{n-1,r-1} - y\delta_{nr} \sum_{r=0}^{n-2} a_{n-1,r}$$

which is the same as the pair of equations

$$(2.2a) \qquad \begin{aligned} a_{n,n-1} &= y \sum_{r=0}^{n-2} a_{n-1,r} \\ a_{nr} - a_{n-1,r} &= a_{n,r-1} + (y - 1)a_{n-1,r-1}, \qquad r = 0, 1, \cdots, n-1. \end{aligned}$$

The second of these along with the boundary condition $a_{nn} = \delta_{n0}$ determines all coefficients a_{nr}. Thus in the first instance $a_{n0} = a_{n-1,0} = \cdots = a_{00} = 1$ while $a_{11} = 0$ and $a_{n1} - a_{n-1,1} = y$ imply $a_{n1} = (n - 1)y$. It follows in succession

Received October 13, 1964.