THE STRUCTURE OF QUASI-OPEN MAPS

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1. Introduction. Light open maps defined on 2-manifolds have been investigated extensively by Whyburn [12], [13] and by Stoilow [10] and have been shown to be of considerable interest because of the close relationship they have with analytic functions. Recently, Church and Hemmingsen [1], [2], [3], have studied the behavior of light open maps defined on n-manifolds. Quasi-open maps have been discussed in [13], [6], and [11]. Quasi-monotone maps, which are the same as quasi-open maps when defined on Peano spaces (see 2.2 below), have been discussed in [12]. These maps have the property that if they are defined on Peano spaces, then their light factors (see 2.2) are open. Therefore, they automatically become of interest since one would expect that, in some respects, they should resemble light open maps.

The purpose of this paper is to initiate a study of the role that open-like maps play in area theory. This is accomplished by determining the structure and behavior of quasi-open maps defined on 2-manifolds and by showing that, from certain points of view, they resemble light open maps. The principal results can be summarized as follows: if f is a quasi-open map defined on a closed 2-manifold, then the middle space of f is locally Euclidean on the complement of a finite set (3.5). From this it follows that the essential multiplicity function (2.4), which in the case of light open maps is the concept of *degree* [13; 90], is constant on the complement of a finite set (3.9). This gives one similarity between quasi-open and light open maps, and it also shows that the Lebesgue area of f is finite and at least as large as the measure of its range. Another similarity is provided by the formula which gives a connection between the singularities of a quasi-open map and the Euler characteristics of its middle space and range (4.3).

2. Definitions and preliminaries.

2.1 Definition. If A is a set, we will let $c(A)$ denote the number (possibly ∞) of points in the set A. If X and Y are topological spaces, $f\colon X \to Y$, and $A \subset X$, then $f \mid A$ will denote the restriction of f to A and *bdry* A will be the topological boundary of A.

2.2 Definition. A mapping $m\colon X \to Y$ is said to be *monotone* if and only if $y \in Y$ implies that $m^{-1}(y)$ is a *continuum*; that is, a compact, connected set. A map $l\colon X \to Y$ is *light* if $l^{-1}(y)$ is totally disconnected for every $y \in Y$. If X is a *Peano space*, that is, a locally connected, connected, compact metric space, then every continuous map $f\colon X \to Y$ has a *monotone-light factorization*

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