THE EXISTENCE OF EXPANSIVE HOMEOMORPHISMS ON MANIFOLDS

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1. **Introduction.** In this paper X will denote a metric space whose metric is d. A homeomorphism, f, of X onto itself is called *expansive* if there exists a positive number c such that for each pair (x, y) of distinct points of X, there exists an integer n for which $d[f^n(x), f^n(y)] > c$. The number c is called an expansive constant for f. Several examples of expansive homeomorphisms are known, and in one of them, due to R. F. Williams [4], the space X is a continuum. W. H. Gottschalk [2; 348] and others have asked whether locally connected spaces or manifolds admit such homeomorphisms. In §3 we construct an expansive homeomorphism of the open cell of even positive dimension, and in §4, one of the r-dimensional torus, if $r \geq 2$. In §2 we prove a theorem which implies all of the known non-existence theorems. In this connection, the reader should see Bryant [1] and Jakobsen and Utz [3].

2. **Non-existence theorems.** If f is a homeomorphism of X onto itself, and x is a point of X, the *α-limit set of x*, denoted by $\alpha(x)$, consists of those points y of X such that $y = \lim_{i\to\infty} f^{n_i}(x)$ for some strictly decreasing sequence of integers n_i. The *ω-limit set of x*, denoted by $\omega(x)$, is similarly defined for strictly increasing sequences n_i. If, for some point x, $\alpha(x)$ and $\omega(x)$ each consist of a single point, we say *x has converging semi-orbits under f.*

THEOREM 1. *If X is compact and f is an expansive homeomorphism, the set of points having converging semi-orbits under f is a countable set.*

Proof. Let c be an expansive constant for f, and say that x is close to y if $d(x, y) < c/2$. Since f is expansive, it has at most finitely many fixed points [1; 388] say $q_1, \cdots, q_k$. Let A denote the set of points having converging semi-orbits, and suppose A is uncountable. If $x \in A$, both $\alpha(x)$ and $\omega(x)$ are fixed points. Let $A(i, j)$ be the set of points for which $\alpha(x) = q_i$ and $\omega(x) = q_j$. Then A is the union of the finitely many sets $A(i, j)$, so that one of these (call it B) is uncountable. For each positive integer N, let $B(N)$ be the collection of points x such that, for $n \geq N$, $f^n(x)$ is close to $\omega(x)$ and $f^{-n}(x)$ is close to $\alpha(x)$. Since B is the union of the sets $B(N)$, one of them, say $B(M)$, must be infinite. Since X is compact, there exist distinct points y and z of $B(M)$ with $d(y, z)$ so small that $f^n(z)$ is close to $f^n(y)$ if $|n| \leq M$. From this remark and the definition of $B(M)$, we conclude that $d[f^n(y), f^n(z)] < c$ for all integers n, contrary to the choice of c. Therefore, A is countable.

COROLLARY 1. *There exists no expansive homeomorphism of an arc.*

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