SECOND ORDER ORDINARY DIFFERENTIAL OPERATORS WITH GENERAL BOUNDARY CONDITIONS

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1. **Introduction.** In the light of the recent work by Sims [5] on singular differential operators of second order and the desire for more general boundary conditions, it seems natural to consider integral boundary conditions. This paper exhibits an operator generated by such a condition, examines its spectrum, resolvent and adjoint. The adjoint operator is of special interest since it is not a purely differential operator. We will use the notation of Sims [5] and Krall [1].

We first require some preliminary results. Let us consider a differential expression of the form $ly = -y'' + q(x)y$ over an interval $[r, b)$ where r is an ordinary point of l and b is a singular point of l. Let $q(x) = q_1(x) + iq_2(x)$ be continuous on $[r, b)$ with $q_2(x) \leq 0$. Let $\lambda = \mu + i\nu$. In a great portion of what follows ν will be restricted to positive values. Finally let $\theta(x, \lambda)$ and $\phi(x, \lambda)$ be solutions of $(l - \lambda)y = 0$ satisfying $\theta(r, \lambda) = 1$, $\theta'(r, \lambda) = 0$, $\phi(r, \lambda) = 0$, $\phi'(r, \lambda) = -1$.

Theorem 1.1. *For all $\nu > 0$ there is a complex function $M(\lambda)$ such that $\psi(x, \lambda) = \theta(x, \lambda) + M(\lambda)\phi(x, \lambda)$ satisfies $(l - \lambda)y = 0$ and is in $L^2(r, b; [1 - q_2(x)]\,dx)$ and hence in $L^2(r, b)$.*

It is further true that

I. $M(\lambda)$ is unique; $\psi(x, \lambda)$ is the only solution of $(l - \lambda)y = 0$ in $L^2(r, b; [1 - q_2(x)]\,dx)$ and $\psi(x, \lambda)$ is the only solution of $(l - \lambda)y = 0$ in $L^2(r, b)$, or

II. $M(\lambda)$ is unique; $\psi(x, \lambda)$ is the only solution of $(l - \lambda)y = 0$ in $L^2(r, b; [1 - q_2(x)]\,dx)$, but every solution of $(l - \lambda)y = 0$ is in $L^2(r, b)$, or

III. All solutions of $(l - \lambda)y = 0$ are in $L^2(r, b; [1 - q_2(x)]\,dx)$ and in $L^2(r, b)$.

In Cases I and II, $M(\lambda)$ is the limit of circles arising from a boundary condition imposed at b' approaching b in $[r, b)$. In Case III the circles approach a limit circle and any $M(\lambda)$ on this circle gives rise to square summable solutions. Hence all are. As a reference to this we site Sims [5] or Krall [1]. We now consider a differential operator in this setting defined by a general boundary condition.

2. **The operator L.** Let D_0 be the set of all functions f defined on $[r, b)$ satisfying

i. f is in $L^2(r, b)$.

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