INTEGRAL FUNCTIONS AND TAUBERIAN THEOREMS

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1. Let $F(z)$ be an integral function of genus zero all of whose zeros are real and negative. In what follows we shall assume that $F(0) = 1$. This seeming restriction will involve no loss of generality as we are interested in certain properties of $F(z)$ as $r = |z|$ approaches infinity. Let

$$n(r) = n(r, F) = \text{number of zeros of } F(z) \text{ in } |z| \leq r.$$

Under these hypotheses we have

$$\log F(r) = \int_0^\infty \log\left(1 + \frac{r}{t}\right) dn(t),$$

i.e.

$$\log F(r) = r \int_0^\infty \frac{n(t)\, dt}{t(t + r)}. \tag{1.1}$$

Also, by analytic continuation,

$$\log F(z) = z \int_0^\infty \frac{n(t)\, dt}{t(t + z)} \tag{1.2}$$

for $|\arg z| < \pi$.

We may write (1.2) as

$$\log |F(re^{i\theta})| = r \int_0^\infty \frac{n(t)(t \cos \theta + r)\, dt}{t(r^2 + t^2 + 2rt \cos \theta)}, \tag{1.3}$$

$$\arg F(re^{i\theta}) = r \sin \theta \int_0^\infty \frac{n(t)\, dt}{r^2 + t^2 + 2rt \cos \theta} \tag{1.4}$$

for $|\theta| < \pi$.

Titchmarsh has pointed out, [12], that theorems relating the growth of $n(r)$ to $\log F(r)$ are closely related to certain Tauberian theorems of Hardy and Littlewood. We have

Theorem A. [6]. *Suppose that* $f(t) \geq 0$. *Suppose also that* $f(t) \in \mathfrak{L}(0, T)$ *for every finite* T *and, that if* $\rho > 0$,

$$\frac{f(t)}{(r + t)^\rho} \in \mathfrak{L}(0, \infty)$$

for some (*and so for all*) $r > 0$. *If, further*

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