THE DERIVATIVE AND THE INTEGRAL OF BANACH-VALUED FUNCTIONS

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In this paper the author considers the differentiability and the integrability (in the sense of Bochner) of an additive set-function of bounded variation. The domain of the set-function is a regular family of measurable sets. No topology is assumed on the measure space. It turns out that if the range of the set-function is a weakly compact Banach space, then the function is weakly differentiable, strongly measurable and Bochner-integrable.

Let (X, Σ, μ) denote a measure space. We shall say that a class $\mathcal{C} \subset \Sigma$ of measurable subsets of X indefinitely covers a subset $Y \subset X$ if for each $y \in Y$, there exists a sequence $A_n \in \mathcal{C}$ such that $y \in A_n$, $n \in \omega$ and $\mu(A_n) \to 0$ as $n \to \infty$. The exterior measure μ_e of a set $A \subset X$ is defined to be $\mu_e(A) = \inf \mu(A')$ over all measurable A' which contain A.

A family $\mathcal{G}$ is called μ-regular if

(i) $\mu_e(\bigcup_{A \in \mathcal{G}} A) < \infty$;
(ii) the set $\rho(A)$ of points outside A and indefinitely covered by subsets A' joint to A has measure zero;
(iii) there exist two numbers a and b $(b > a > 1)$ such that for every $A \in \mathcal{G}$, $\mu_e(\Omega(A)) < b\mu(A)$ where

$$\Omega(A) = \{\cup A' : A' \in \mathcal{G},\ A' \cap A \neq \phi,\ \mu(A') < a\mu(A)\}.$$

Theorem 1 (Vitali-Denjoy). *Let $\Delta(\mathcal{G})$ [abbreviated Δ] denote the set of points indefinitely covered by the regular family $\mathcal{G}$. Then*

(i) *there exists a sequence $\{A_n\}$ of disjoint sets in $\mathcal{G}$ such that*
$$\mu(\Delta - \Delta \cap \bigcup_{n \in \omega} A_n) = 0;$$
(ii) *for every $\epsilon > 0$ a sequence $\{A_n\}$ of disjoint sets in $\mathcal{G}$ can be chosen such that* $\mu(\bigcup_{n \in \omega} A_n) < \mu(\Delta) + \epsilon$.

The proof of this theorem can be found in [3].

Definitions A. Let $\mathcal{G} = \{A\}$ be a regular family of sets and let $\{\epsilon_k\}$ be a decreasing sequence of positive real numbers converging to zero. Let $\mathcal{G}^k$ denote the family of sets A in $\mathcal{G}$ for which $\mu(A) < \epsilon_k$. Since $\mathcal{G}^k$ indefinitely covers Δ, by Theorem 1 there exists a denumerable subfamily $\{A_i^k\}$ of $\mathcal{G}^k$ which covers Δ a.e., i.e. $\mu(\Delta - \Delta \cap \bigcup_{i \in \omega} A_i^k) = 0$. Let $R^k = \Delta - \Delta \cap \bigcup_{i \in \omega} A_i^k$ and $R = \bigcup_{k \in \omega} R^k$. $\mu(R) = 0$ since $\mu(R^k) = 0$. Now let $\Delta' = \Delta - R$, $\eta_i^i = A_i^i \cap \Delta'$

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