TRANSLATION INVARIANT OPERATORS IN L^p

By Alessandro Figà-Talamanca

1. Introduction. Let G be a locally compact Abelian group with Haar measure dx. We shall be concerned with the space M_p of bounded operators T on $L^p(G)(1 < p < \infty)$ which commute with translations; that is, $\tau_x T = T\tau_x$, for all $x \in G$, where $\tau_x f(y) = f(x + y)$. The space M_p is an algebra of operators on L^p which is closed in the weak operator topology. We prove (Theorem 1, below) that M_p is the closure in the weak operator topology of the span of the translation operators. This is accomplished by proving that M_p is the dual space of a Banach space A_p of continuous functions on G. The space A_p consists of functions of the type $\sum_1^\infty f_i^* g_i$ with $f_i \in L^p$, $g_i \in L^q(1/p + 1/q = 1)$ and $\sum_1^\infty \|f_i\|_p \|g_i\|_q < \infty$. In §3 an analogous theorem is proved for a compact group in which the hypothesis of commutativity is dropped: that is, the algebra of operators on L^p which commute with right translations is the closure of the span of left translations and vice versa.

Results contained in this paper appeared in part in a dissertation submitted in partial satisfaction of the requirements for the Ph.D. degree at the University of California, Los Angeles, which was prepared under the direction of Professor Philip C. Curtis, Jr. to whom I am greatly indebted for advice and assistance. Theorem 1 was announced in [3]. The claim made in Remark 6 of [3] of the validity of Theorem 1 for general unimodular groups was based on an incorrect proof. The problem in the context of non-compact, non-commutative groups remains open.

2. The commutative case. In what follows p will satisfy $1 < p < \infty$ and q will be defined by $1/p + 1/q = 1$. Let $T \in M_p$ be a translation invariant operator. If we denote by C_{00} the space of continuous functions with compact support on G, it is not difficult to see that $Tf*g = T(f*g) = f*Tg$ for every $f, g \in C_{00}$. Indeed an application of Fubini's Theorem will show that

$$\int_G Tf * g(x)k(x)\, dx = \int_G f * g(x)T'k(x)\, dx = \int_G T(f * g)(x)k(x)\, dx,$$

where k is any element of L^q and T' is the (Banach space) adjoint of T. Thus $T(f*g) = Tf*g$ and, reversing the role of f and g, $T(f*g) = f*Tg$. Now

$$|f * Tg(0)| = |Tf * g(0)| \leq \|T\|_{M_p} \|f\|_p \|g\|_q ,$$

by Hölder's inequality, therefore the restriction of T to C_{00} can be extended as a bounded linear operator to L^q, with $\|T\|_{M_q} \leq \|T\|_{M_p}$. Reversing the roles of

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