PROJECTIONS ON CONTINUOUS FUNCTION SPACES

By Richard Arens

1. Introduction. In 1940, R. S. Phillips (see the bibliography) showed that there was no bounded projection of (m) onto (c), while in 1944 A. Sobczyk showed that if S was a separable (closed) subspace of (m), including (c), then there did exist a bounded projection of S on (c). These results, and also those of B. Grünbaum are compatible with, and did suggest to P. C. Curtis Jr. and the author, the specious

Conjecture. *If $\mathcal{C}(X)$ is isometric with a closed subspace C of a separable Banach space S, then there is a bounded projection of S on C.*

This conjecture shall now be destroyed. In fact, there is (see 3.5 below) a countable closed bounded subset X of the line such that $S = \mathcal{C}(X, \mathbf{R})$, which is obviously separable, contains a closed subspace isometric to $\mathcal{C}(Y, \mathbf{R})$, but there is no bounded projection of the former on the latter.

In this example, the space Y is an identification space of X, arising from a decomposition D of X. The projection problem is just the problem of projecting $\mathcal{C}(X, \mathbf{R})$ onto the D-functions, that is to say, the functions constant on the sets of D. We relate this to a new concept, the derived decomposition D'. Our analysis shows that Sobczyk's projection exists, not only because of the separability, but because $D^{(n)} = 0$ for some n. We show that (2.7 below) when $D^{(n)} = 0$, then there exists a projection of bound at most 3^n, which can be improved to $4n - 1$ when X is compact.

When the decomposition D has exactly one set Z which is plural (i.e., has more than one point) and if Z contains exactly n limit points of the complement, then there is a projection of bound $3 - 2/n$ (see 2.4) and no projection of lesser bound (see 3.1).

For all these results we suppose that X and X/D are metric, but not always compact.

2. The construction of bounded projections. When X and Y are topological spaces, then $\mathcal{C}(X, Y)$ denotes the class of continuous functions $f : X \to Y$. In the present section, we shall let $\mathcal{C}(X)$ stand for $\mathcal{C}(X, L)$ where L is some normed linear space over the reals $\mathbf{R}$. For example L might be $\mathbf{R}$ or the complex field $\mathbf{C}$. The L will be fixed for the entire discussion.

In any case, $\mathcal{C}(X)$ may be "normed",

2.01 $$||f|| = \sup_{x \in X} ||f(x)||$$

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