AN ITERATED LOGARITHM LAW FOR LOCAL TIME

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1. **Introduction.** P. Erdös posed to the author the problem of finding strong limit laws for the maximal multiplicity in n steps of a simple random walk in one dimension (as $n \to \infty$). The analogous problem for dimension two and higher had been solved by Erdös and Taylor [3]. We consider here the continuous time analogue, namely we derive limit laws for the maximum (over the space variable) of the local time of Brownian motion. This problem can be very well treated with the help of two recent papers by Knight [4] and Ray [7]. The corresponding result for random walks (Theorem 3) is stated without proof even though its proof is not a mere "translation to the discrete case" of the proofs of Theorem 1 and 2.

Throughout this note we use the following notation. $X(t, \omega)$ is a standard Brownian motion ($X(0, \omega) = 0$ with probability 1, $E\,X(t, \omega) = 0$, $E\,X^2(t, \omega) = t$) and $f(x, t) = f(x, t, \omega)$ is its local time, i.e. the continuous (jointly in x and t) function which satisfies for each measurable set E

$$\int_E f(x, t, \omega)\, dx = \int_0^t \chi_E(X(\tau, \omega))\, d\tau \tag{1.1}$$

where $\chi_E(\cdot)$ is the characteristic function of the set E. The existence of a continuous local time was shown by Trotter [8]. Finally

$$f(t) = f(t, \omega) = \sup_x f(x, t, \omega).$$

When convenient we shall not write the argument ω.

THEOREM 1.

$$\limsup_{t\to\infty} \frac{f(t, \omega)}{\sqrt{t \log\log t}} = \limsup_{t\to\infty} \frac{f(0, t, \omega)}{\sqrt{t \log\log t}} = \sqrt{2}$$

with probability 1.

THEOREM 2. *There exists a constant γ such that*

$$\frac{q_0}{2} \le \gamma \le \frac{q_0^2}{\sqrt{2}}$$

and

$$\liminf_{t\to\infty} \frac{f(t, \omega)\sqrt{\log\log t}}{\sqrt{t}} = \gamma$$

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