GENERATING FUNCTIONS FOR POWERS OF A CERTAIN GENERALISED SEQUENCE OF NUMBERS

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1. Introduction. Two fundamental sequences in the theory of second-order recurrences are the Fibonacci sequence $\{f_n\}$ and the Lucas sequence $\{a_n\}$ defined by:

$$\begin{array}{lllllllll} n & :0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{array}$$

$$\{f_n\} : 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad \cdots \tag{1}$$

$$\{a_n\} : 2 \quad 1 \quad 3 \quad 4 \quad 7 \quad 11 \quad 18 \quad \cdots \tag{2}$$

where

$$f_n = f_{n-1} + f_{n-2} = \frac{\alpha_1^{n+1} - \beta_1^{n+1}}{\alpha_1 - \beta_1} \qquad (n \geq 2) \tag{3}$$

$$a_n = a_{n-1} + a_{n-2} = \alpha_1^n + \beta_1^n \tag{4}$$

in which α_1, β_1 are the roots of $x^2 - x - 1 = 0$, so that

$$\alpha_1 = \frac{1+\sqrt{5}}{2}, \beta_1 = \frac{1-\sqrt{5}}{2}, \alpha_1 + \beta_1 = 1, \alpha_1\beta_1 = -1, \alpha_1 - \beta_1 = \sqrt{5}. \tag{5}$$

Classical extensions of these are the sequences $\{u_n\}$, $\{v_n\}$ defined by:

$$\{u_n\} \equiv \{u_n(p, q)\} : u_0 = 1, u_1 = p, u_n = pu_{n-1} - qu_{n-2} \qquad (n \geq 2) \tag{6}$$

$$\{v_n\} \equiv \{v_n(p, q)\} : v_0 = 2, v_1 = p, v_n = pv_{n-1} - qv_{n-2} \tag{7}$$

with $p^2 \neq 4q$ and p, q arbitrary integers. (Comments on the degenerate case $p^2 = 4q$ are made towards the end of §4.)

Another extension is defined in [4], with some obvious notational alterations, and for arbitrary integers r, s, by:

$$\{h_n\} \equiv \{h_n(r, s)\}: h_0 = r, h_1 = r + s, h_n = h_{n-1} + h_{n-2} = rf_n + sf_{n-1} (n \geq 2). \tag{8}$$

Sequences (6) and (7) have long exercised interest; see for instance, Bessel-Hagen [1], Lucas [8] and Tagiuri [10], and, for historical details, Dickson [3]. Important particular cases of them are [8],

the Fermat sequences

$$\begin{cases} \{u_n(3, 2)\} \equiv \{2^{n+1} - 1\} : 1 \ 3 \ 7 \ 15 \ 31 \ \cdots \\ \{v_n(3, 2)\} \equiv \{2^n + 1\} \quad : 2 \ 3 \ 5 \ \ 9 \ 17 \ \cdots \end{cases}$$

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