THE SUMMABILITY OF LAGUERRE SERIES

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1. **Introduction.** Let ϕ be Lebesgue integrable over $[0, R]$ for every $R > 0$ and for $\alpha > -1$ let

$$\Gamma(\alpha + 1)\binom{n + \alpha}{n}a_n = \int_0^\infty e^{-t}t^\alpha L_n^{(\alpha)}(t)\phi(t)\,dt \tag{1.1}$$

exist for $n = 0, 1, \cdots$ where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial of degree n defined by

$$(n!)e^{-x}x^\alpha L_n^{(\alpha)}(x) = D_x^n(e^{-x}x^{n+\alpha}). \tag{1.2}$$

Then we write

$$\phi(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \tag{1.3}$$

and the series is called the Fourier-Laguerre series of ϕ. The convergence of this series has been carefully studied [6; 244]. In addition, the Abel sum was considered by Caton and Hille [1; 227] and by Wigert [9]. In all of these results the restriction on ϕ at $t = \infty$ is severe. For example, $\phi(t) = e^{\beta t}$ for $\frac{1}{2} < \beta < 1$ does not satisfy the conditions cited by these authors although the coefficients $\{a_n\}$ of (1.1) do exist. It is the main purpose of this paper to exhibit a summability method which will essentially close this gap. This result is in §3. Also in §3, we apply this result to the theory of the Laplace transform. §2 contains the results needed for the main theorem in §3.

2. **Some lemmas.** For simplicity we will confine ourselves to the case $\alpha = 0$. A series $\sum_{n=0}^\infty u_n$ is summable $(B, 1)$ to s if

$$\int_0^\infty e^{-t}\left[\sum_{n=0}^{\infty} u_n t^n/n!\right] dt = s.$$

This is a Borel method and its theory is well known, [4; 182]. Previous work with Hermite polynomials and the relationship between Laguerre and Hermite polynomials have suggested that this is the desired summability method.

The following functions will be important.

$$S(x, y, t) = \sum_{n=0}^{\infty} L_n(x)L_n(y)t^n/n! \tag{2.1}$$

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