HELLY'S THEOREM AND MINIMA OF CONVEX FUNCTIONS

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1. Introduction. The object of this paper is to prove an existence theorem for solutions to a very general class of constrained and unconstrained minimization problems involving convex functions on R^n. This theorem is in effect an extension of the classical theorem of Helly, according to which an infinite collection of compact convex sets in R^n has a nonempty intersection if every $n + 1$ of the sets have a point in common. (For the general literature on Helly's theorem see the expository article by Danzer, Grünbaum and Klee in *Convexity*, Proceedings of the Symposium in Pure Mathematics, vol. VI, American Mathematical Society, 1963.)

The idea of extending Helly's theorem to convex functions is not new; such extensions have been given by Bohnenblust, Karlin and Shapley [2; 185] (discussed also in [13]) and by Fenchel in his 1953 lecture notes [7; 96–101]. Both of these, however, are limited essentially to collections of convex functions on a bounded convex set. Our theorem does not have this limitation, and hence it can be used both in the compact case and in the theory of convex programming, where compactness is usually too severe a restriction. It implies, for instance, that a polynomial convex function achieves a minimum on any polyhedral convex set where it is bounded below, a result obtained in the quadratic case by Frank and Wolfe [8]. Yet at the same time it contains, in a direct way, a new generalization of Helly's Theorem in which the sets and their intersections can sometimes all be unbounded.

Our principal device is to replace compactness, wherever this might otherwise be necessary, by "asymptotic regularity conditions" which restrict behavior along certain infinite rays which might be present. This was suggested by Fenchel's work with the asymptotic cones of convex sets [7, 42–44 and 99–101].

Besides applying the existence theorem to ordinary convex programs, we shall derive from it results in the theory of inequalities and Lagrange multipliers complementary to those in [6]. A new general version of von Neumann's minimax theorem, not requiring compactness, will also be deduced.

2. Existence theorem. Throughout this paper P will denote a non-empty polyhedral convex set in R^n, i.e. a set which can be represented as the intersection of finitely many closed half-spaces. The choices of P we have most in mind are: R^n itself, the "non-negative orthant" of R^n, the unit simplex, the product of n closed intervals of R, or some combination of these, such as the

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