THE MODULE TYPE OF HOMOMORPHIC IMAGES

By W. G. Leavitt

Received January 20, 1964.

1. Introduction. In a recent paper [2] the concept of the module type of a ring K (to be designated $t(K)$) was introduced. (Throughout this paper any ring considered will be assumed to have an identity.) It was shown [2; Theorem 2; 115] that if $K \to K'$ is a unit-preserving homomorphism (for example, if it is an epimorphism) then $t(K') \leq t(K)$ under a natural ordering of types. The question naturally arises as to what types the homomorphic images of a ring may have. In particular, does there exist a ring K with images of all types $\leq t(K)$? The construction given below shows that the answer is affirmative for rings of arbitrary type. More generally, for any type b there exists a ring K for which $t(K) = b$ such that for an arbitrary sequence of types $b > a_1 > \cdots > a_s$ there is a set of rings $\{K_i\}$ with $t(K_i) = a_i$ and with a sequence of epimorphisms $K \to K_1 \to \cdots \to K_s$. Moreover, K may itself be chosen to be the final image of such a sequence having any set of types $\geq t(K)$. For the purpose of this construction a division ring is used. However, any of a fairly wide class of rings will do as well, and such rings are considered in the final section.

2. Preliminary results. The concept of module type may be somewhat generalized as follows: Let $\mathcal{R}$ be a class of rings closed under direct sums and homomorphisms (that is, direct sums and homomorphic images of members of $\mathcal{R}$ remain in $\mathcal{R}$). A map $\tau : \mathcal{R} \to \mathcal{S}$ of $\mathcal{R}$ into a lattice $\mathcal{S}$ is called a *type map* if it satisfies conditions: For all $R_1, R_2 \in \mathcal{R}$

(i) When $R_1 \to R_2$ is an epimorphism $\tau(R_2) \leq \tau(R_1)$.

(ii) $\tau(R_1 \oplus R_2) = \tau(R_1) \cup \tau(R_2)$.

Lemma 1. *Let I and J be ideals of a ring $K \in \mathcal{R}$. If $J \subseteq I$, then $\tau(K/I) \leq \tau(K/J)$.*

Proof. From (i), since K/I is a homomorphic image of K/J.

Theorem 1. *Let I and J be ideals of a ring $K \in \mathcal{R}$. If I is maximal, then $\tau(K/J \cap I) = \tau(K/J) \cup \tau(K/I)$.*

Proof. Let $H = I \cap J$. If $J \subseteq I$, then it follows from Lemma 1 that $\tau(K/I) \leq \tau(K/J)$. Then $\tau(K/H) = \tau(K/J) = \tau(K/J) \cup \tau(K/I)$. Thus assume $J \not\subseteq I$. By maximality, $K = J + I$ so that $K/J \cong I/H$ and $K/I \cong J/H$. Thus $K/H \cong K/J \oplus K/I$, and the theorem follows from (ii).

Corollary 1. *In the ring $K \in \mathcal{R}$ let J be an ideal and $\{I_i\}$, $(i = 1, \cdots, n)$ a set of maximal ideals. Writing $H = \bigcap_{i=1}^{n} I_i$, then*