PROPERTIES OF SOLUTION-PRESERVING OPERATORS

BY H. J. BIESTERFELDT, JR.

Sheffer and Krall have established [2] a representation of certain solution-preserving linear partial differential operators. We study the ring of such operators and characterize its units, showing that in general the operators are onto but not $1 - 1$.

We use the notation $L(w) = w_{xy} - w_x - w_y$. Λ is the class of analytic solutions to $L(w) = 0$ (the L-equation). We deal with linear partial differential operators $T : \Lambda \to \Lambda$ with analytic coefficients, and agree that operators will be reduced to standard form by elimination of the mixed partials using $L(w) = 0$. The following theorem is a rephrasing of a theorem of Sheffer and Krall.

THEOREM 0. *Let $T : \Lambda \to \Lambda$ be an operator as above. Then $T(w)$ has a representation*

$$T(w) = \sum a_{j,p,q} H^j D_x^p D_y^q(w)$$

where $j + p + q \leq n$, $pq = 0$, p and q are equal to zero or one. n is the order of the standard form of T, $H(w) = (x - y)w - xw_x + yw_y$, and $a_{j,p,q}$ are constants.

From this theorem we can deduce analogous results in terms of other variables x', y' linear in x, y. For $L = (D_x^2 + D_x) + i(D_y^2 + D_y)$, the operator $H(w) = (x + iy)w + 2iyw_x + 2xw_y$ provides the nontrivial part of an operator of order 1. In addition, there are the differentiations. Combinations and composition of these provide all operators under consideration.

We first find the kernel of the operator H given by Theorem 0. Using Lagrange's method and solving locally for w we get $w = e^{x+y}f(xy)$. If now $L(w) = 0$, we have

$$e^{x+y}[f + yf' + xf' + xyf'' + f' - f - yf' - f - xf'] = 0$$

for all x and y. Hence with $u = xy$,

$$uf'' - f + f' = 0. \tag{1}$$

Any function $f(u)$ satisfying (1) gives us a solution $w \in \Lambda$ and conversely. Thus H-operators are not generally $1 - 1$. However they are onto, $\Lambda \to \Lambda$. For simplicity we first consider the operator $H(w) = (x + iy)w + 2iyw_x + 2xw_y$, using the second equation above. Let $w \in \Lambda$. We solve $H(W) = w$ for $W \in \Lambda$.

For this we find two functions $f(x)$ and $g(x)$ satisfying

$$xf(x) + 2xg(x) = w(x, 0)$$

$$if(x) + xg(x) + 2if'(x) + 2x[- g(x) - if'(x) - if''(x)] = w_y(x, 0).$$

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