EXISTENCE OF SOLUTIONS OF NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN L^2.

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In 1957 Sims [2] extended the limit point-limit circle argument (of Weyl [4], Titchmarsh [3], and Coddington and Levinson [1]) which shows that under certain conditions the equation

$$-\psi'' + (q(x) - \lambda)\psi = 0 \tag{1}$$

has a square summable solution, to a non-self adjoint problem. The purpose of this paper is to show that the limit point-limit circle argument yields a natural proof that the nonhomogeneous equation

$$-u'' + (q(x) - \lambda)u = K(x) \tag{2}$$

also has square summable solutions provided $K(x)$ is square summable. A necessary and sufficient condition is derived.

Since the method of proof is closely related to the original limit point-limit circle arguments, the method of Sims [2] will first be sketched.

Let r be interior to (a, b); let $q(x) = q_1(x) + iq_2(x)$ be continuous on (a, b); let $\lambda = \mu + i\nu$ and $\alpha = \alpha_1 + i\alpha_2$. Further assume that $q_2(x) \leq 0$, $\nu > 0$, $\alpha_2 \leq 0$. Let $\phi(x)$ and $\theta(x)$ be solutions of (1) satisfying $\phi(r) = \sin\alpha$, $\phi'(r) = -\cos\alpha$, $\theta(r) = \cos\alpha$, $\theta'(r) = \sin\alpha$. Note that for all x, $W[\phi(x), \theta(x)] = \phi(x)\theta'(x) - \phi'(x)\theta(x) \equiv 1$.

Throughout the remainder of this paper λ will be fixed with $\nu = \text{Im}(\lambda) > 0$.

We will be concerned with two types of Hilbert spaces: (i) $L^2(dx)$ or (ii) $L^2([1 - q_2(x)]\,dx)$. Note that if a function is in any of the spaces (ii), it is also in the corresponding space (i). Further note that since $\nu > 0$, $q_2(x) \leq 0$, $\int [\nu - q_2(x)]\,|f|^2\,dx < \infty$ if and only if $\int [1 - q_2(x)]\,|f|^2\,dx < \infty$.

Theorem 1 (Sims). *For all $\nu > 0$, there is a complex function M $(=M(\lambda))$ such that $\psi_1(x) = \theta(x) + M\phi(x)$ satisfies* (1) *and is in $L^2(r, b; [1 - q_2(x)]\,dx)$ and hence is in $L^2(r, b)$.*

Let z be an arbitrary complex number, and let $\psi(x, l, z) = \theta(x) + l(z, b')\phi(x)$ be a solution of (1) satisfying

$$z\psi(b', l, z) + \psi'(b', l, z) = 0 \tag{3}$$

for b' in (r, b). Then $l(z, b')$ is determined for all complex boundary conditions (3) by

$$l(z, b') = -(\theta(b')z + \theta'(b'))/(\phi(b')z + \phi'(b'))$$

Received October 18, 1963. This is a part of a Ph.D. dissertation written at the University of Virginia. The author wishes to thank Professors E. J. McShane and Marvin Rosenblum for their assistance.