BASIC ALGEBRAS OF ALGEBRAS WITH UNIQUE MINIMAL FAITHFUL REPRESENTATIONS

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1. Introduction. Let $\mathfrak{A}$ be a finite dimensional algebra with identity over a field ϕ. In an earlier paper [6] the author studied algebras having unique minimal faithful representations (UMFR algebras). The classes of algebras studied were those subclasses of the UMFR algebras which contain the generalized uniserial algebras as a subclass. For any algebra $\mathfrak{A}$ we can choose a subalgebra $\mathfrak{A}^*$ of $\mathfrak{A}$ called a basic algebra for $\mathfrak{A}$. The purpose of the present paper is to study the basic algebras of algebras in the various subclasses of the UMFR algebras. In particular, we are interested in determining what can be said about $\mathfrak{A}^*$ if $\mathfrak{A}$ is a certain type and what can be said about $\mathfrak{A}$ if $\mathfrak{A}^*$ is a certain type. For example, it is shown (Proposition 6) that if $\mathfrak{A}^*$ is type A, then both $\mathfrak{A}^*$ and $\mathfrak{A}$ are type AC, and it is shown (Proposition 11) that if $\mathfrak{A}$ is UMFR, then there exists an algebra $\mathfrak{B}$ of type A such that $\mathfrak{B}^* \simeq \mathfrak{A}^*$.

§2 contains properties of idempotents and matrix units and §3 contains the definition of and properties of the basic algebras. §4 contains the definitions of the classes of algebras to be studied and a reformulation of the results of the previous paper [loc. cit.]. In §5 the relations between the algebras and basic algebras in the various classes are obtained. For any class X of algebras let $X^* = \{\mathfrak{A}^* \mid \mathfrak{A} \in X\}$ and let $X^0 = \{\mathfrak{A} \mid \mathfrak{A}^* \in X\}$. The results of §5 are summarized in terms of the X^* and X^0 classes. §6 contains examples to show that certain classes are distinct.

2. Idempotents and matrix units. Let $\mathfrak{A}$ be a finite dimensional algebra with identity element over a field ϕ. If $e \in \mathfrak{A}$, then e is an *idempotent* if $e^2 = e \neq 0$. If e and f are idempotents, then e is *orthogonal* to f if $ef = fe = 0$. An idempotent e is *primitive* if it is not the sum of two orthogonal idempotents. Idempotents e and f of $\mathfrak{A}$ are *isomorphic*, $e \simeq f$, if $\exists\ a, b \in \mathfrak{A}$ such that $ab = e$ and $ba = f$. If $e \simeq f$, then the a and b may be chosen so that $a \in e\mathfrak{A}f$ and $b \in f\mathfrak{A}e$. (For proofs of this and other results stated here see [1], [2] and [5].) For idempotents e and f the following are equivalent:

$$(1) \qquad e \simeq f; \qquad \mathfrak{A}e \simeq \mathfrak{A}f; \qquad e\mathfrak{A} \simeq f\mathfrak{A};$$

where $\mathfrak{A}e \simeq \mathfrak{A}f$ means that $\mathfrak{A}e$ and $\mathfrak{A}f$ are isomorphic as left $\mathfrak{A}$-modules.

Let

$$(2) \qquad 1 = \sum_{i=1}^{n} \sum_{j=1}^{f(i)} e_{ij}$$

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