NEW FOUNDATIONS AND THE AXIOM OF COUNTING

By Steven Orey

Introduction. We shall be concerned with the system NF, the New Foundations of Quine [5]. Working with this system Rosser [6] noticed that a certain obvious proposition of intuitive set theory did not appear to be provable in NF, and considered adding it to the system as a new axiom, the so called *axiom of counting*:

$$(n) : n \,\varepsilon\, \mathrm{Nn}. \supset . \{m \mid m \,\varepsilon\, \mathrm{Nn}.\, 0 < m \leq n\} \,\varepsilon\, n.$$

In this paper we show that the axiom of counting cannot be proved to be equivalent to a stratified formula in NF unless it is disprovable; *a fortiori* the axiom of counting can not be proved in NF, if NF is consistent. A number of related results are established. One of these is the non-finite axiomatizability of a system intimately connected with NF.

We make essential use of the connections between NF and type theory discovered by Specker [8], [9]; this will be discussed in §1. We shall use without explicit mention the fact that the "axiom of infinity", $\sim \Lambda \,\varepsilon\, \mathrm{Nn}$, is provable in NF, established in Specker [7]. We shall also use the result of Hailperin [2] that NF can be finitely axiomatized.

For any system S which we introduce we always assume some fixed Gödel numbering. In Peano Arithmetic (the system Z of [3]) we can then discuss the syntax of S. In particular Con_S is to be a natural assertion of consistency of S in Peano Arithmetic. (It was pointed out in [1] that the "usual" construction of the assertion of consistency of S in Peano Arithmetic involves some arbitrariness: for if $A(S)$ is the set of Gödel numbers of axioms of S one must construct a formula $\alpha(x)$ numeralwise representing this set. If $A(S)$ is a recursively enumerable such a formula, $\alpha(x)$ will exist, but it will not be unique. When we refer to a *natural assertion of consistency* we are demanding that $\alpha(x)$ must be an *RE*-formula, in the terminology of [1]. This still does not determine a unique α, and that is why we simply take Con_S to be *some* natural assertion of consistency.)

If S is a formal system and Φ a recursive collection of sentences of S, $S \cup \Phi$ is to be the system obtained by adding all members of Φ as new axioms. We write NFC for NF $\cup$ {axiom of counting}. We will establish that $\mathrm{Con}_{\mathrm{NF}}$ is provable in NFC. (After finding this result we tried to "improve" it to show $\mathrm{Con}_{\mathrm{NF}}$ provable in NF. Once we thought we succeeded; our sincere thanks are due to Professor Dana Scott for spotting the error.)

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