SEPARABILITY AND METRIZABILITY IN POINTWISE PARACOMPACT MOORE SPACES

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For connected metric spaces, P. Alexandroff [1] has shown that local separability implies separability, and F. B. Jones [8] has shown that local peripheral separability together with local connectedness implies separability. Recently L. B. Treybig [17] and J. H. Roberts [14] have obtained results complementing Alexandroff's, and P. Roy [15] has obtained an analogous result in the direction of Jones' theorem. The proofs of all of these theorems rely heavily on the triangle inequality. However, the proofs can be based on pointwise paracompactness instead. In fact all of the theorems generalize to pointwise paracompact Moore spaces. In this direction, D. R. Traylor [16] has shown that a locally separable, pointwise paracompact Moore space is metrizable; also he has shown that a pointwise paracompact, connected Moore space is metrizable if it satisfies conditions similar to the hypotheses of Jones' theorem; thus, in either case, by Alexandroff's or Jones's theorem respectively, the space is separable if connected. In this paper, Roberts' and Roy's theorems are generalized to pointwise paracompact Moore spaces, Jones' theorem is extended to pointwise paracompact topological spaces, and a closely related theorem of Jones [10] is generalized from metric to pointwise paracompact Moore spaces (with a considerable simplification of the proof). Corresponding metrization theorems follow as immediate corollaries, and two other related metrization theorems are obtained.

The following lemma, which is useful in subsequent theorems, is established by an elementary argument similar to that for Theorem 2 of [11] or as a corollary to Theorem 3 of [7].

Lemma 0. *In a pointwise paracompact Moore space, every separable set is hereditarily separable (and each of its uncountable subsets has a limit point).*

Theorem 1 is a generalization of Roberts' theorem in [14]. The proof follows somewhat the same outline as Roberts'. The following two lemmas of Roberts', which can be easily seen to hold for a space X satisfying the hypotheses of Theorem 1, are stated for convenience.

Lemma 1. *If x, y and z are three points of X, then there exists a closed and connected point set W containing y and z but not x.*

Lemma 2. *If $M \subset X$ and M is closed, connected and non-degenerate, then every component of $X - M$ is open and has at least two limit points in M.*

Theorem 1. *Suppose that* (1) *X is a connected pointwise paracompact Moore space,* (2) *X contains no cut points and* (3) *for all p, q in X, $p \neq q$, and every open*

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