THE EXPONENTS OF INCIDENCE MATRICES

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1. Introduction. This paper is concerned with the multiplicative properties of matrices whose entries are non-negative real numbers. Such matrices are said to be *non-negative*. They arise naturally in many ways, e.g. the transition matrices of stochastic processes; the adjacency and incidence matrices of graphs, projective planes, statistical designs; matrices connected with network flows and other combinatorial problems. Because of these applications, as well as their own interest, these matrices merit an independent investigation. The classical work on these matrices was carried out by Perron, Frobenius and Wielandt in [11], [7] and [14]. More recently Pták and Sedlacek [12], [13], Holladay and Varga [8], Perkins [10] and the present authors [5], [6], have studied such matrices from the combinatorial point of view.

In the classical theory of non-negative matrices, a matrix A is said to be *reducible* if there exists a permutation matrix P such that $P^{-1}AP = \begin{bmatrix} B & 0 \\ D & C \end{bmatrix}$ where B and C are square matrices and 0 is a zero matrix. If no such permutation matrix P exists then A is said to be *irreducible*. If every entry of A is positive, we say that A is *positive* and write $A > 0$. If A is irreducible, it may happen that a power of A is positive. In this case A is said to be *primitive* and the least power of A which is positive is called the *exponent* of A and is denoted by $\gamma(A)$. It is well known that a non-negative matrix is irreducible if and only if the directed graph of the matrix is strongly connected. It is well known also, as remarked in [5], that a non-negative irreducible matrix is primitive if and only if the directed graph of the matrix is such that the greatest common divisor of the circuit lengths is 1. A basic theorem concerning primitive matrices states that A is primitive if and only if A has a unique positive characteristic root α of multiplicity 1 such that if β is any other characteristic root of A, then $|\beta| < \alpha$. At a Combinatorial meeting held at the RAND Corporation in the summer of 1961, O. Taussky used this theorem to prove that if A is any N by N incidence matrix of a finite projective plane, then A^{N-1} is positive, and then posed the problem of determining the exponent of A. In this paper it is shown that $A^4 > 0$ and that there exist permutation matrices P and Q such that $(PAQ)^3$ has at least one zero entry.

In this paper, as in [5], [6], [8], [10], [12] and [13], the method of determining primitivity is purely combinatorial. This is as it should be for if A and B are non-negative matrices, the zero or non-zero character of an entry in the product

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