COMPACT DUAL SEMIGROUPS WITHOUT NILPOTENT IDEALS

By Katsumi Numakura

1. Introduction. By a *topological semigroup* $S(\mathfrak{I})$ we shall mean an abstract semigroup S which is also a Hausdorff space (in the topology $\mathfrak{I}$), such that ab is a continuous function of a and b, a, b being elements of S. The abstract semigroup S is said to the *underlying semigroup* of $S(\mathfrak{I})$. If S contains a zero, i.e. an element 0 such that $x0 = 0 = 0x$ for all $x \varepsilon S$, $S(\mathfrak{I})$ is said to be a *topological semigroup with* 0. In this paper we treat only topological semigroups with 0, therefore we shall use the term "semigroup" to denote topological semigroup with 0. We shall usually write S instead of $S(\mathfrak{I})$ when there is no danger of ambiguity.

Let L be a closed left ideal of a semigroup S and $r(L)$ the right annihilator of L; $r(L)$ is a closed right ideal of S. The semigroups to be studied in this paper, which we call "dual semigroups", are those semigroups in which the mapping $L \rightarrow r(L)$ sets up a one-to-one correspondence between the closed left ideals and the closed right ideals of S. This notion was first introduced by Baer [1] and Kaplansky [8] for rings. In these papers the introductory notions of duality use only the multiplicative properties of the ring under study. Therefore it seems natural to apply this hypothesis to the theory of semigroups. However, the only work dealing with dual semigroups, of which the author is aware, is that of Schwarz. In [13] he has discussed discrete dual semigroups, and obtained a number of interesting results. In particular, he has determined the structure of discrete [semi-] simple dual semigroups possessing minimal ideals.

The purpose of the present paper is to develop the structure theory of compact dual semigroups which are simple or semisimple. After elementary preparations in §2, we shall prove, in §3, semisimplicity of certain dual semigroups (Theorem 1). We shall also give, in the section, a decomposition theorem for compact or discrete dual semigroups without nilpotent ideals (Theorem 2). The result is an improvement of a theorem of Schwarz. In §4, the structure theory of compact simple dual semigroups will be treated. We shall show there that a compact simple dual semigroup is represented as a Rees $n \times n$ matrix semigroup over a compact group with 0 (Theorem 9). In the last section, §5, we shall study topological properties of compact semisimple dual semigroups, and show necessary and sufficient conditions that a compact semisimple semigroup be dual (Theorems 11 and 15).

For most of the terminology and the algebraic properties on semigroups used in this paper we refer to Clifford and Preston [4]. It is noticed, however, that

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