THE INFINITE SUM OF CLOSED SUBSPACES OF AN F-SPACE

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Introduction. The first part of this paper treats various conditions under which the sum of a collection of closed subspaces of a complete linear metric space, F-space, will be closed. The second part is a discussion of F-spaces which can be written as the sum of an infinite sequence of closed subspaces. It will be seen that certain theorems which are true about Schauder bases will also hold for sequences of closed subspaces whose sum is closed and which are s.l.i. (defined below). For this reason McArthur [10] was led to call such a sequence of closed subspaces a Schauder decomposition.

Definition. Let X be an F-space; and let $\{M_i : i = 1, 2, \cdots\}$ be a sequence of subspaces of X. For brevity we shall write (M_i) when no ambiguity results.

a) The symbol $\sum_{i=1}^{\infty} M_i$ represents the set

$$\left\{x \in X : x = \sum_{i=1}^{\infty} m_i \ , \ m_i \in M_i\right\}.$$

b) The sequence (M_i) is *strongly linearly independent* (s.l.i.) if and only if $0 = \sum_{i=1}^{\infty} m_i$, $m_i \in M_i$ implies $m_i = 0$ for each i.

c) The symbol $(w) \sum_{i=1}^{\infty} M_i$ represents the set

$$\left\{x \in X : x^*(x) = \sum_{i=1}^{\infty} x^*(m_i);\ m_i \in M_i \quad \text{for each} \quad x^* \in X^*\right\}.$$

d) The sequence (M_i) is *weakly linearly independent* (w.l.i.) if and only if for each $x^* \in X^*$, $0 = \sum_{i=1}^{\infty} x^*(m_i)$ with $m_i \in M_i$ implies $m_i = 0$ for each i.

If $\{x_s : s \in S\}$ is an infinite set of elements of X, the series $\sum_{s \in S} x_s$ is unordered convergent to $x_0 \in X$ if and only if letting $\mathfrak{F}$ be the collection of finite nonempty subsets of S, there corresponds to each neighborhood V of 0 in X and $F_V \in \mathfrak{F}$ such that if $F \in \mathfrak{F}$ and $F \supset F_V$, then $(x_0 - \sum_{s \in F} x_s) \in V$. The symbol $\sum_{s \in S}$ will always represent such an unordered sum.

e) Let $\{M_s : s \in S\}$ be a collection of subspaces of X which are indexed by S. We shall usually write (M_s). Define $\sum_{s \in S} M_s$ to be the set $\{x \in X : x = \sum_{s \in S} x_s ;\ x_s \in M_s\}$.

f) The collection (M_s) is *unordered strongly linearly independent* (u.s.l.i.) if and only if $0 = \sum_{s \in S} m_s$; $m_s \in M_s$ implies $m_s = 0$ for each s.

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