SUMS OF PROJECTIONS

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Following the notation employed by N. Dunford and co-workers in their study of spectral-type operators, we define a projection on a Hilbert space H, to be a bounded idempotent transformation of H into itself, (not necessarily self-adjoint). In a finite dimensional space when projections sum to a projection, they are pairwise disjoint; and the sum of projections is never a non-zero negative operator.

These last remarks can be interpreted as saying that projections globally try to be "positive", on a finite dimensional space. In this paper we will exhibit the surprising change which takes place when the underlying Hilbert space is infinite dimensional. Our main result states that any self-adjoint operator is the sum of four projections. We will first quote some well-known results to give perspective to that which follows.

Theorem A. *If $E_1, \cdots, E_n$ are projections on a finite dimensional Hilbert space $\bar{H}$ and $E_1 + \cdots + E_n = I$, then $E_iE_j = 0$ $i, j = 1, \cdots, n$ $i \neq j$, i.e. the E_1's are pairwise disjoint.*

The use of the trace affords a quick and easy proof which obviously will not generalize to the infinite dimensional case. Note that if we have $E_1 + \cdots + E_n = E_{n+1}$ where $E_1, \cdots, E_{n+1}$ are projections, we can reduce this to the case where $E_1 + \cdots + E_k = I$ by adding the projection $I - E_{n+1}$ to both sides of the initial equation. From here on H will denote a separable Hilbert space, which is infinite dimensional.

Theorem B. *If $E_1, \cdots, E_n$ are self-adjoint projections on H, with $E_1 + \cdots + E_n = I$; then the E_i are pairwise disjoint.*

Proof. Let $x_1 \in$ Range E_1, that is $E_1x_1 = x_1$; then $(E_1 + \cdots + E_n)x_1 = Ix_1$ implies $(E_2 + \cdots + E_n)x_1 = 0$. Thus $0 = (E_2 + \cdots + E_n x_1, x_1) = \sum_{i=2}^n ||E_ix_1||^2$; hence, $E_ix_1 = 0$ for $i = 2, \cdots, n$. By considerations of symmetry it is now easy to see that the E_i's are indeed pairwise disjoint.

Theorem C. *If $E_1, \cdots, E_n$ are projections on H, where $E_1 + \cdots + E_n = I$ and $E_iE_j = E_jE_i$ for $i, j = 1, \cdots, n$; then the E_i's are pairwise disjoint.*

Proof. By a well-known result (2) there exists a positive invertible operator Q such that QE_iQ^{-1} is a self-adjoint projection for $i = 1, \cdots, n$. This reduces the proof to that of Theorem B.

We now proceed to the main part of the paper and first examine what happens when a small number of projections sum to a multiple of the identity.

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