A FUNCTIONAL-DIFFERENCE EQUATION

By L. Carlitz

1. J. A. Morrison [1] has considered the functional-difference equation

$$(1.1)\qquad (x-\alpha)(\alpha-\beta)^{n-1}g_n(x) = \alpha(x-\beta)^n g_{n-1}(\alpha) - x(\alpha-\beta)^n g_{n-1}(x)(n \geq 1),$$

where $g_0(x) = 1$. It follows from (1.1) that, for $n \geq 1$, $g_n(x)$ is a polynomial of degree $n-1$ in x with coefficients depending on α, β. Morrison has proved that

$$(1.2)\qquad g_n(\alpha) = \sum_{r=0}^{n-1} \binom{n}{r}\binom{n-1}{r}\beta^{n-r}\alpha^r/(r+1) = \beta^n F(-n, -n+1; 2; \alpha/\beta).$$

It follows easily from (1.1) that

$$(1.3)\qquad G_n(x) = G_n(x, \alpha, \beta) = (\alpha-\beta)^{n-1}g_n(x) \qquad (n \geq 1)$$

is a homogeneous polynomial of degree $2n-1$ in x, α, β. It is of some interest to find an explicit formula for $G_n(x)$. Now, as Morrison has observed,

$$(-1)^n(x-\alpha)^n g_n(x)$$
$$= (\alpha-\beta)^n x^n + \alpha \sum_{r=1}^{n} (-1)^r x^{n-r}(x-\alpha)^{r-1}(x-\beta)^r(\alpha-\beta)^{n-2r+1}g_{n-1}(\alpha).$$

Hence, expanding the right member in powers of $x-\alpha$, we obtain a polynomial expression for $g_n(x)$. However, it seems better to proceed differently.

We may put

$$(1.4)\qquad G_n(x) = \sum_{r=0}^{n-1} A_r^{(n)}(x-\beta)^r(\alpha-\beta)^{n-r-1} \qquad (n \geq 1)$$

where $A_r^{(n)} = A_r^{(n)}(\alpha, \beta)$ is homogeneous of degree n in α, β. Also put

$$(1.5)\qquad G(t) = \sum_{n=1}^{\infty} G_n(x)t^n.$$

It is clear from (1.1) that

$$g_n(\beta) = \beta g_{n-1}(\beta) \qquad (n \geq 1).$$

Since $g_0(\beta) = 1$, it follows that

$$(1.6)\qquad g_n(\beta) = \beta^n \qquad (n = 0, 1, 2, \cdots).$$

Comparison with (1.3) and (1.4) gives

$$(1.7)\qquad A_0^{(n)} = \beta^n.$$

Received July 17, 1963. Supported in part by NSF grant G 16485.