A CERTAIN FUNCTIONAL-DIFFERENCE EQUATION

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1. Introduction. In the study by J. P. Runyon [1] of a queueing system in which a group of servers handles traffic from two sources, one of which is preferred over the other, there arose the functional-difference equation

$$(1.1)\qquad \begin{aligned}(\alpha - x)(\beta - \alpha)^{n-1} g_n(x) &= [\alpha(\beta - x)^n g_{n-1}(\alpha) - x(\beta - \alpha)^n g_{n-1}(x)], \quad n \geq 1;\\ g_0(x) &= 1.\end{aligned}$$

Here $0 < \alpha < \beta$. It follows by induction that $g_n(x)$ is a polynomial of degree $(n - 1)$ in x, with coefficients depending on α and β, for $n \geq 1$. Of particular interest, however, is the value of $g_n(\alpha)$. (It is of interest to mention that W. S. Brown has since used his computer program for the symbolic manipulation of rational functions of several variables to calculate the first fifteen $g_n(x)$, and the corresponding $g_n(\alpha)$.) Now Runyon calculated the first several polynomials $g_n(x)$, and thence the corresponding $g_n(\alpha)$, from which he conjectured that

$$(1.2)\qquad g_n(\alpha) = \sum_{r=0}^{n-1} \binom{n-1}{r}\binom{n}{r} \beta^{n-r}\alpha^r/(r + 1) = \beta^n F[-n, -(n - 1); 2; \alpha/\beta],$$

where $F(a, b; c; z)$ denotes a hypergeometric function. (J. Riordan has pointed out that these same polynomials arise in the study of the moments of the delay function for last come first served service by a simple trunk group (with Poisson input and exponential holding times).) We proceed to establish this conjecture.

2. A transformation of the equation. Now (1.1) may be written in the form

$$(2.1)\qquad \left\{\left[-\frac{(\alpha - x)}{x(\beta - \alpha)}\right]^n g_n(x) - \left[-\frac{(\alpha - x)}{x(\beta - \alpha)}\right]^{n-1} g_{n-1}(x)\right\} = (-1)^n \alpha x^{-n}(\alpha - x)^{n-1}(\beta - x)^n(\beta - \alpha)^{1-2n} g_{n-1}(\alpha).$$

Summing from $n = 1$ to $n = m$, and remembering that $g_0(x) = 1$, we obtain

$$(2.2)\qquad \left[-\frac{(\alpha - x)}{x(\beta - \alpha)}\right]^m g_m(x) = \left[1 + \sum_{n=1}^{m} (-1)^n \alpha x^{-n}(\alpha - x)^{n-1}(\beta - x)^n(\beta - \alpha)^{1-2n} g_{n-1}(\alpha)\right].$$

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