COMBINATORIAL REMARKS ON PARTITIONS OF A MULTIPARTITE NUMBER

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1. **Introduction.** A partition of an integer m is uniquely determined by stating the multiplicity of each summand occurring in it. Thus the representation $15 = 1 + 1 + 1 + 2 + 5 + 5$ is equivalent to the function v given by $v(0) = 0, v(1) = 3, v(2) = 1, v(3) = 0, v(4) = 0, v(5) = 2, v(6) = \cdots = v(15) = 0$. Evidently partitions of m into nonnegative parts correspond to functions defined on $\{0, 1, \cdots, m\}$ and partitions of m into nonzero (i.e. positive) parts to functions defined on $\{1, 2, \cdots, m\}$. This definition has an obvious generalization to partitions of multipartite numbers where it simplifies combinatorial matters.

2. **Partitions considered as functions and as lattice points of convex solids.** Lower case Latin letters denote real numbers or functions, Latin capital letters sets. $A \times B$ is the Cartesian product of A and B. $A \simeq B$ means that A and B have the same number of elements. $\emptyset$ is the empty set. $F_n = \{\beta = b_1, \cdots, b_n) : b_j$ is a nonnegative integer for each j, $1 \leq j \leq n\}$ is the set of n-partite numbers. By definition $|\beta| = b_1 + \cdots + b_n$ is the weight of β and $G_n = \{\mu = (m_1, \cdots, m_n) \varepsilon F_n : 2 \leq |\mu|\}$. $\theta = (0, \cdots 0)$ is the zero element of F_n. Let $e_{jj} = 1$ for each j, $e_{ji} = 0$ if $j \neq i$. Then $\epsilon(j) = (e_{j1}, \cdots, e_{jn})$ for each j, $1 \leq j \leq n$. $\beta \leq \mu$ means $b_j \leq m_j$ for each j, $1 \leq j \leq n$. $\beta < \mu$ means $\beta \leq \mu$ and $\beta \neq \mu$.

Lower case Greek letters always represent elements of F_n and when one occurs as an index in a sum, product or union it runs through all values in F_n obeying stated restrictions, if any. If $T \subset F_n$ then $B(T) = \{v : T \to F_1\}$ is the set of nonnegative integer valued functions with domain T. If $T = \{\xi : \theta < \xi \leq \mu\}$ then $B(T)$ is written $B(\mu)$. If $T = \{\xi : \xi \leq \mu\}$ then $B(T)$ is written $B_\theta(\mu)$.

Definition. Let $r \varepsilon F_1$, Let $T = T(\mu) = \{\xi \varepsilon G_n : \xi \leq \mu\}$. If S is a finite subset of G_n, let $S^* = S \cup \{\epsilon(1), \epsilon(2), \cdots, \epsilon(n)\}$.

$$P_S(\mu) = \{v \varepsilon B(S^*): \sum_{\xi \varepsilon S^*} \xi v(\xi) = \mu\};$$

$$P(\mu) = P_T(\mu);$$

$$U_r(\mu) = \{v \varepsilon B(\mu) : \sum_{\theta < \xi \leq \mu} v(\xi) = r\};$$

$$P_r(\mu) = P(\mu) \cap U_r(\mu);$$

$$Q_r(\mu) = \{v \varepsilon B_\theta(\mu) : \sum_{\xi \leq \mu} \xi v(\xi) = \mu, \quad \sum_{\xi \leq \mu} v(\xi) = r\}.$$

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