NOTE CONCERNING A WILD SPHERE OF BING

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1. **Introduction.** A closed set X in E^3 is *tame* if there is a homeomorphism h of E^3 onto itself such that $h(X)$ is a polyhedron. A closed set that is not tame is called *wild*. An example has been constructed by R. H. Bing of a 2-sphere S in E^3 which is wild, yet each of whose arcs is tame [3]. One complementary domain of E^3-S, however, is not simply connected; in fact, if D is any disk in S, then there exists a simple closed curve J in E^3-S such that J cannot be shrunk to a point in E^3-D. Thus, in [3], Bing raises the following question:

Is a 2-sphere in E^3 tame if each of its arcs is tame and each of its complementary domains is simply connected?

The main purpose of this note is to answer this question in the negative. The counterexample, which we call S', is obtained by merely a slight alteration in the construction of S. The difficulty, however, lies in the fact that the methods of [3] cannot be used to establish that every arc in S' is tame.

In §2, a brief review of the construction of S is given, followed by a description of the modification necessary to produce S'. In §3, an alternative proof to that of [3] is given to show that all arcs in S are tame. The alternative proof is adjusted slightly in §4 to show that all arcs in S' are tame. Indeed, this argument establishes even more: Every closed nowhere dense subset of S (or S') is "tame" in that it lies on a tame 2-sphere. The fifth section shows that not only are both complementary domains of S' simply connected, but each is even homeomorphic to an open 3-cell (assuming the point ∞ has been added to E^3, of course).

The construction and methods discussed here have recently been put to use in many different ways, such as to yield uncountably many embeddings of the 2-sphere in E^3 [1], to construct a decomposition of E^3 into points and pointlike arcs [8], and to construct a simple closed curve which pierces no disk, yet lies on a disk [7].

Wherever possible, notation used will correspond to that of [3].

2. **The examples.** We begin by describing the 2-sphere S of [3]. A tame 2-sphere is decomposed into disks E_1, E_2, $\cdots$, E_{15} such that adjacent E_i's have an edge in common (E_1 and E_{15} also share a common edge). Each E_i is thickened and an "eye bolt" added to give a solid torus T_i, with the loop of T_i circling the stem of T_{i+1} (the loop of T_{15} circles the stem of T_1). See Figure 1.

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