ENTIRE SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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1. Let $P(\xi)$ be a non-constant polynomial in $\xi = (\xi_1, \cdots, \xi_n)$ and let $u(x)$ be a weak solution (i.e., a solution in the sense of distributions) of

$$P(iD_x)u(x) = 0 \quad \text{for} \quad x \in R^n, \tag{1}$$

where $x = (x_1, \cdots, x_n)$, $D_x = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$, and R^n is the real n-dimensional Euclidean space. We denote by L^p the set of all measurable functions $v(x)$ in R^n satisfying: $\int_{R^n} |v(x)|^p \, dx < \infty$. We also write a distribution w as $w(x)$ and denote by $(w(x), \varphi(x))$ the application of w to a test function φ.

THEOREM 1. *If $u \in L^2$ and u satisfies* (1), *then $u(x) = 0$ almost everywhere.*

Proof. Considering u as a tempered distribution (see [6]), its Fourier transform $\tilde{u}$ satisfies

$$P(\xi)\tilde{u}(\xi) = 0.$$

(For functions, $\tilde{\varphi}(\xi) = \int e^{ix\cdot\xi} \varphi(x)dx$.) Hence the support of $\tilde{u}$ is contained in the manifold $N(P) = \{\xi; \xi \in R^n \text{ and } P(\xi) = 0\}$. The distribution $\tilde{u}$ coincide with the distribution defined by the classical Fourier transform, say $\mathfrak{F}u$, of u. Hence the support of the distribution defined by $\mathfrak{F}u$ is also contained in $N(P)$. Since $\mathfrak{F}u \in L^2$, $\mathfrak{F}u = 0$ almost everywhere. Hence $u = 0$ almost everywhere.

2. We are interested in the following problem: Given $P(\xi)$, determine the largest p_0 such that whenever $u(x)$ satisfies (1) and $u \in L^p$ for some $2 \leq p \leq p_0$ (or $2 \leq p \leq p_0$), $u(x) = 0$ almost everywhere.

It then follows that if $u(x) = O(|x|^{-m})$, as $|x| \to \infty$, for some $m > n/p_0$ then $u(x) = 0$ almost everywhere.

The case of elliptic P is of particular interest since we then obtain an extension of the classical Liouville theorem. It may be recalled that if $P(\xi) \neq 0$ for all real $\xi \neq 0$ (P is not necessarily elliptic) then the only tempered distributions which are solutions of (1) are polynomials and, therefore, in this case $p_0 = \infty$. We shall now consider a class of elliptic operators P for which $N(P)$ consists of hyperspheres.

THEOREM 2. *Let $P(\xi) = Q(|\xi|^2)$, Q being a polynomial. If u is a solution of* (1) *and $u \in L^p$ for some $2 \leq p \leq 2n/(n-1)$, then $u(x) \equiv 0$. There exist nonzero solutions of* (1) *which belong to L^p for any $p > 2n/(n-1)$.*

Theorem 2 is related to Theorem 4 in [2] and can be proved using the results

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