CYCLICLY RELATED DIFFERENTIAL EQUATIONS

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1. **Introduction.** It was shown [1] that the system of differential equations

$$y_i' = \sum_{k=1}^{n} A_k y_{i+m+hk}\,, \qquad \begin{cases} i = 1, \cdots, n; \quad y_{i+n} \equiv y_i\,; \\ A_k \quad \text{integrable on } [a, b], \end{cases}$$

where h and m are integers, first investigated by W. M. Whyburn [3], could be equivalently replaced by the matrix differential equation

$$Y' = AY, \qquad A \text{ a circulant,} \tag{1}$$

and solved explicitly.

This paper shows that the fundamental matrix Y of (1) such that $Y(a) = I$, the identity matrix, is a circulant, implying that one column determines the matrix. Also considered is the matrix differential equation

$$Y' = AEY, \tag{2}$$

where A is a circulant and the $n \times n$ matrix $E = (e_{ij})$ is defined by

$$e_{ij} = \begin{cases} 1 & \text{if} \quad i + j = n, 2n \\ 0 & \text{elsewhere,} \end{cases} \tag{3}$$

and a fundamental matrix exhibited. The paper in turn considers the interrelation between (1) and (2) as well as the matrix equation $Y' = (A + BE)Y$, where A and B are circulants.

2. **Circulants and anticirculants.** Let σ denote the algebra of real valued functions which are continuous on $[a, b]$ and let $a_1, \cdots, a_n$ belong to σ with the agreement that $a_n = a_0$. (This is a restriction for simplicity. Extensions of the theorems to more general functions will be apparent.)

Definition. An $n \times n$ matrix $A = (a_{ij})$ is called i) *a circulant* [2] *with elements* $a_1, \cdots, a_n$ (or simply a circulant) if, and only if, $a_{ij} = a_{j-i+1 \bmod n}$, ii) *an anticirculant with elements* $a_1, \cdots, a_n$ (or simply an anticirculant) if, and only if, $a_{ij} = a_{j+i-1 \bmod n}$.

The following theorems are easily proven.

Theorem 1. a) *A is a circulant* [2] *with elements $a_1, \cdots, a_n$ if, and only if, $A = \sum_k a_k F^{k-1}$, where $F = (f_{ij})$ is the matrix defined by*

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