A CONVOLUTION PRODUCT FOR DISCRETE FUNCTION THEORY

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1. Introduction. This paper is concerned with complex-valued functions defined at the points of the complex plane whose coordinates are integers. Isaacs [6], Ferrand [5] and Duffin [4] have developed function theories for functions of this type. The function theory used here is based on the work of these authors.

The points of the complex plane with integer coordinates form a lattice which breaks up the plane into unit squares. A function f is said to be discrete analytic on one of these squares, if the difference quotient across one diagonal is equal to the difference quotient across the other diagonal,

$$\frac{f(z+1+i)-f(z)}{1+i}=\frac{f(z+i)-f(z+1)}{i-1}.$$

The work of Isaacs and Duffin shows that it is difficult to define a product for two discrete analytic functions which preserves discrete analyticity. They investigate operators which correspond to multiplication of continuous analytic functions, but these operators preserve analyticity only if one of the factors is a polynomial. Here, a different approach is taken to the problem of multiplication. Using the definition of a discrete line integral given by Duffin [4], a "convolution product" of two discrete functions is defined, which is commutative, associative and distributive; and if the functions involved are discrete analytic, then so is the product. This product is a summation over a chain of lattice points, which resembles the convolution integral appearing in the theory of the Laplace transform. Two other similar line integrals given in reference [4] are shown to have similar properties to the convolution product. These products are termed hyper-convolution products.

Using the machinery developed, concerning the convolution product, a theory similar to that of Volterra integral equations is developed. Discrete derivative equations are also introduced and related to the discrete Volterra integral equation theory.

The convolution product is also related to a discrete Laplace transform. This Laplace transform yields theorems analogous to the continuous case. For example, the inversion formula for these "Laplace transforms" is given which can be used to continue discrete functions defined on the x-axis to discrete analytic functions in the complex plane.

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