ISOTOPY INVARIANTS OF TREES

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1. **Introduction.** The deleted product space X^* of a topological space X is the subset of the cartesian product of X with itself consisting of pairs of distinct points. If X and Y are topological spaces and $f : X \to Y$ is a continuous map, let X_f^* be the space which is the inverse image of Y^* in the map $f \times f : X \times X \to Y \times Y$. In [1], Brahana asks the following question: *What maps f are such that there is a homotopy equivalence between X_f^* and X^*?* The objective of this paper is to provide a partial answer to this question if X and Y are trees (finite, contractible, 1-dimensional polyhedra). The results indicate that it will be very difficult to provide a complete answer.

We use the integers as the coefficient group for the homology groups, and the group $H_0(K)$ is the direct sum of $m - 1$ groups each isomorphic to the integers, where m is the number of components of K.

In [2], Eilenberg shows that X^* is connected if X is not an arc. In [4], the author shows that if X is a tree, then $H_2(X^*) = 0$ and $H_1(X^*)$ is a free abelian group. A formula for calculating the number of generators of $H_1(X^*)$ is given.

In §3, we are concerned with the connectivity of X_f^*. In §4, using the results mentioned above, we show that, for a certain type of map, $H_2(X_f^*) = 0$ and $H_1(X_f^*)$ is a free abelian group, and we compute the exact number of generators of $H_1(X_f^*)$.

2. **Preliminary results.** If X is a finite polyhedron, let $P(X^*) = \cup\{r \times s \mid r$ and s are simplexes of X and $r \cap s = \phi\}$. In [3; 348–352], Hu proves that X^* is homotopically equivalent to $P(X^*)$. Shapiro's proof [5; 257] of this theorem is incorrect because his retraction is not continuous. In order to see this, consider the following example.

Let K be the complex consisting of three vertices, v_0, v_1, v_2, and two 1-simplexes, $\langle v_0, v_1\rangle$, $\langle v_1, v_2\rangle$. Let $p = (1/2)v_0 + (1/2)v_1$ and $q = (1/2)v_1 + (1/2)v_2$. Then, using Shapiro's notation, $\beta(p, q) = v_0$ and $\beta(q, p) = v_2$. Let $q' = (1/2 - \epsilon)v_1 + (1/2 + \epsilon)v_2$, where ϵ is an arbitrarily small positive number. Then $\beta(p, q') = p$ and $\beta(q', p) = v_2$. Therefore $h(p, q, 1) = (v_0, v_2)$ and $h(p, q', 1) = (p, v_2)$. Hence h_1 is not continuous at (p, q).

If X and Y are finite polyhedra and $f : X \to Y$ is a simplicial map, let

$$P(X_f^*) = \cup \{r \times s \mid r \text{ and } s \text{ are simplexes of } X \text{ and } f(r) \cap f(s) = \phi\}.$$

Theorem 1. *If X and Y are finite polyhedra and $f : X \to Y$ is a simplicial map, then X_f^* is homotopically equivalent to $P(X_f^*)$.*

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