TRANSLATION INVARIANT SPACES WITH ZERO-FREE SPECTRA

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I. **Introduction.** Let $\tilde{H}$ denote the linear space of all functions $\tilde{h}(x)$ in $L_2(-\infty, \infty)$ which vanish for negative values of their argument. We say that a subspace $\tilde{L}$ of $\tilde{H}$ is *left translation invariant* if $\tilde{l}(x) \in \tilde{L}$ implies that the projection of $\tilde{l}(x + \tau)$ onto $\tilde{H}$ belongs to $\tilde{L}$ for all positive τ. A subspace $\tilde{R}$ of $\tilde{H}$ is called *right translation invariant* if the right translate $\tilde{r}(x - \tau)$ of every element $\tilde{r}(x) \in \tilde{R}$ belongs to $\tilde{R}$ for all positive τ. The orthogonal complement with respect to $\tilde{H}$ of a left translation invariant space is right translation invariant, and conversely.

Now take $\tilde{L}$ to be any closed left translation invariant subspace of $\tilde{H}$ and let $T : \tilde{L} \rightarrow \tilde{L}$ be the one-sided shift operator defined by

$$(T\tilde{l})(x) = \begin{cases} \tilde{l}(x+1) & \text{if} \quad x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A description of the non-zero elements of $\sigma(T)$, the spectrum of T, in terms of an analytic function characterizing $\tilde{L}$ was given in an earlier paper [5], but the problem of deciding when $\sigma(T)$ contains the origin was left unsolved. Our aim here is to settle this question and, in addition, to find a bound for $\|T^{-1}\|$ when the origin lies outside of $\sigma(T)$. The estimate obtained for $\|T^{-1}\|$ has some applications in communication theory which we plan to explore in a forthcoming paper.

II. **Spectral analysis.** Let H denote the space of functions $h(s)$ which are the Fourier transforms of functions in $\tilde{H}$, i. e.

$$F(\tilde{h}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \tilde{h}(x)\, dx = h(s).$$

The space H is characterized by the one-sided

Paley-Wiener Theorem. *Every function h in H can be extended as a regular analytic function into the upper half-plane in such a way that*

$$\int_{-\infty}^{\infty} h(s + it) h^*(s + it)\, ds \leq \textit{constant}$$

for all positive values of t. Conversely, the restriction to the real axis of any such function belongs to H [7].

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