REARRANGEMENTS OF SQUARE MATRICES WITH NON-NEGATIVE ELEMENTS

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Introduction. In their book on inequalities, Hardy, Littlewood and Pólya define rearrangements of an ordered set (a) of n numbers $a_1, \cdots, a_n$ in the following way [5, Chapter X]: *Let $\varphi(j)$, $j = 1, \cdots, n$ be a permutation function, i.e., a function which takes each of the values $1, \cdots, n$ just once when j varies through the same aggregate of values. If*

$$a_{\varphi(j)} = a'_j, \qquad j = 1, \cdots, n,$$

then the set $(a') = (a'_1, \cdots, a'_n)$ is called a rearrangement of the given set $(a) = (a_1, \cdots, a_n)$. We may look upon (a) and (a') as vectors; two vectors are mutual rearrangements if they have the same unordered set of components. Here we use the term "rearrangement of a square matrix of order n" in the same sense, i.e. with regard to the unordered set of its n^2 elements. Given a set σ of n^2 numbers we obtain a set $\mathfrak{M}$ of $(n^2)!$ square matrices of order n and if the given numbers are pairwise different then the matrices M of $\mathfrak{M}$ will also be pairwise different. $\mathfrak{M}$ is thus the set of all rearrangements of any of its matrices.

Some rearrangements of a given matrix $M = (m_{ij})$, $i, j = 1, \cdots, n$, are often considered: the transpose $M^T = (m_{ij}^T)$, $m_{ij}^T = m_{ji}$ and all the permutations of M. $M' = (m'_{ij})$ is a permutation of M if $m'_{ij} = m_{\varphi(i)\varphi(j)}$ where $\varphi(i)$, $i = 1, \cdots, n$, is a permutation function. A permutation is therefore a rearrangement of the rows combined with the same rearrangement of the columns [4; 50]. We call these two kinds of rearrangements and their combinations trivial rearrangements and we say that M_1 and M_2 are essentially different if M_2 is a rearrangement of M_1 but not a trivial one. If the n^2 numbers of σ are pairwise different, then $\mathfrak{M}$ splits into $(n^2)!/2n!$ subsets $\mathfrak{M}_i$; each $\mathfrak{M}_i$ contains $2(n!)$ mutual trivial rearrangements and two matrices belonging to different subsets are essentially different.

Throughout this paper we assume that the n^2 elements of σ are real and non-negative. In §1 we consider for any given $\mathfrak{M}$ the extrema of $||M^2||$, $M \in \mathfrak{M}$. Here $||M^2|| = \sum k_{ij}$ where $M^2 = K = (k_{ij})$ (and $m_{ij} \geq 0$ implies $k_{ij} \geq 0$). This norm of the square is invariant under trivial rearrangements while $||M||$ and any other Hölder norm $||M||_p$ [7] of M itself is invariant under all rearrangements and hence constant for the whole set $\mathfrak{M}$. If M_1 and M_2 are essentially different, then, in general, $||M_1^2|| \neq ||M_2^2||$. But to find the extrema of $||M^2||$, $M \in \mathfrak{M}$, we do not have to compute this norm of the square for all essentially different matrices of $\mathfrak{M}$. Indeed, denoting by $\mathfrak{A}^+$ the subset of $\mathfrak{M}$ on which $||M^2||$ is maximal and by $\mathfrak{C}^+$ the subset of all matrices of $\mathfrak{M}$ for which the

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