SOME THEOREMS ON ČEBYŠEV APPROXIMATION

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1. **Introduction and preliminary remarks.** The purpose of this paper is to establish some theorems on the approximation of continuous functions in the uniform (Čebyšev) norm. For our proofs we require certain known results from the theory of Čebyšev approximation. These results may be found, with proofs and references to the literature, in [1]. (In [1] the underlying space X is taken to be Euclidean, but the proofs are valid without change in the generality discussed below.)

Let X be a compact Hausdorff space and $C(X)$ the Banach space of continuous functions on X normed by $\| f \| = \max | f(x) |$. Let Φ be a k-dimensional subspace of $C(X)$. Then $\phi^* \varepsilon \Phi$ is a *best approximation* to f if $\| f - \phi^* \| = \min \| f - \phi \|$, the min (which is attained) being over all $\phi \varepsilon \Phi$. Given any finite point set $x_1, \cdots, x_n$ and n unit vectors $\epsilon_1, \cdots, \epsilon_n$ the couples (x_i, ϵ_i) are said to form a *signature* which we denote by Σ. A subset of these couples is a *subsignature* of Σ. A signature Σ is *extremal* (relative to Φ) if there exist positive numbers p_i such that $\sum p_i \bar{\epsilon}_i \phi(x_i) = 0$ for all $\phi \varepsilon \Phi$. (Notationally it is convenient (as is done in 2 below) to write these finite sums as integrals with respect to discrete measures.) We have the following theorem: If $\| f - \phi \| = m$ and $f(x_i) - \phi(x_i) = \epsilon_i m$ where the pairs (x_i, ϵ_i) form an extremal signature, then ϕ is a best approximation to f. Conversely if ϕ is a best approximation to f, and $\| f - \phi \| = m$, there exists an extremal signature (x_i, ϵ_i) such that $f(x_i) - \phi(x_i) = \epsilon_i m$. (All functions here are complex-valued.)

A *set of uniqueness* for Φ is a set $X_0 \subset X$ such that $\phi \varepsilon \Phi$, $\phi(x) = 0$ for $x \varepsilon X_0$ implies $\phi \equiv 0$. An extremal signature for which the x_i form a set of uniqueness we call a *strong extremal signature*. If in the above theorem $f(x_i) - \phi(x_i) = \epsilon_i m$ where (x_i, ϵ_i) form a strong extremal signature, ϕ is the *unique* best approximation to f. Φ is a *Čebyšev system* if $\phi \varepsilon \Phi$, $\phi(x) = 0$ at k distinct points implies $\phi \equiv 0$. In this case the best approximation is always unique. When X is a real interval and Φ is a real Čebyšev system, every set of $k + 1$ distinct points endowed with alternating signs is a (strong) extremal signature.

2. **Čebyšev approximation in a Cartesian product space.** (*Added in proof.* The authors have learned of some related work of Charles Lawson in a 1961 Dissertation at U. C. L. A. Lawson obtains our Theorem 1 in the special case that Φ and Ψ are Čebyšev systems on real intervals. He also obtains some uniqueness results, and has some interesting observations on the problem treated here.) 2.1 Let X and Y be compact Hausdorff spaces. Let Φ be an m-dimensional subspace of $C(X)$, Ψ an n-dimensional subspace of $C(Y)$. Let $\phi_i(x)$, $i = 1, \cdots,$

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