ROTATION-INVARIANT ALGEBRAS ON THE n-SPHERE

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Let X be the n-sphere and let $C(X)$ be the continuous complex functions on X. One can exhibit a host of rotation-invariant subspaces of $C(X)$. Indeed for each integer $d \geq 0$ one has the invariant space $P_d(X)$ of homogeneous harmonic polynomials of degree d in $n + 1$ variables; and by using an appropriate summability method one can form arbitrary direct sums of such $P_d(X)$. In the present paper we want to point out that invariant sub*algebras* are remarkably less numerous.

THEOREM. *For X = the n-sphere, $n \geq 2$, the algebra $C(X)$ contains exactly one non-trivial subalgebra invariant under rotation. This subalgebra consists of functions that identify antipodal points, and hence is essentially $C(Y)$ with Y = projective n-space.*

Here and elsewhere in this paper all "subalgebras" and "subspaces" are taken to be closed. The "trivial" invariant subalgebras are the whole algebra $C(X)$, the constants, and $\{0\}$.

The proof of the theorem is little more than a conjunction of four facts: (1) The invariant subspaces of $C(X)$ are in one-one correspondence with the invariant subspaces of $P(X)$ = the algebraic direct sum of the $P_d(X)$. (2) The invariant symmetric subalgebras of $C(X)$ are in one-one correspondence with the invariant fiberings of X. (3) All invariant subspaces of $P(X)$ are symmetirc. (4) There is exactly one non-trivial invariant fibering of X.

By a "fibering" of X we mean simply a collection of disjoint closed subsets such that the union of the collection is X and the induced quotient space Y is Hausdorff. The fibering is "invariant" if each rotation takes each fiber onto another fiber, hence can be regarded as a motion on Y.

The four steps of the argument outlined above will in fact go through in a larger context than that of the theorem. One can generalize both the X and the $C(X)$.

With regard to X, the first two steps of the argument are valid for an arbitrary homogeneous X acted on continuously by a compact group U. In step (1), $P(X)$ becomes the algebra of "spherical polynomials", i.e. functions p having only finitely many linearly independent translates p^u. Moreover, the last two steps are valid with appropriate changes for most of the compact symmetric Riemannian manifolds in Elie. Cartan's celebrated list (in particular, for all 2-point-homogeneous spaces of dimension $n \geq 2$) since by a curious coincidence of nomenclature most of these manifolds are also symmetric in the sense of having all the invariant subspaces of $P(X)$ closed under complex conjugation.

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