DERIVATIONS OF LIE ALGEBRAS III

To Professor E. Bompiani, belatedly, for his Scientific Jubilee

By G. Leger

1. **Introduction.** If V is a finite dimensional vector space over a field K of characteristic 0 and if T is any K-endomorphism of V, then T may be written uniquely in the form $T = S + N$ where $[S, N] = SN - NS = 0$, S is semi-simple (i.e. has simple elementary divisors) and where N is nilpotent. If A is an algebraic Lie algebra of K-endomorphisms of V and if T is in A, then both S and N belong to A. A subalgebra of A will be called a *toroidal subalgebra of* A if it is abelian and consists of semi-simple endomorphisms. In [11] Mostow has the following results:

I. Any two maximal fully reducible subalgebras of a linear Lie algebra L are conjugate under an inner automorphism from the radical of $[L, L]$.

II. If A is a maximal toroidal subalgebra of the linear Lie algebra L, then there exists a maximal semi-simple subalgebra S of L such that $A = A \cap S + A \cap R$ (R = radical of L) with 1) $A \cap S$ a Cartan subalgebra of S, 2) $A \cap R$ is a maximal toroidal subalgebra of R and 3) $[S, A \cap R] = 0$.

Thus if L is a Lie algebra with derivation algebra $D(L)$, the dimension of a maximal toroidal subalgebra of $D(L)$ is an invariant of L which we denote by $t(L)$.

The central motif of the paper is the use of properties of the derivation algebra to obtain structural properties of the Lie algebra itself. In §2 we shall be concerned with results linking $t(L)$ with the structure of L while in the final §3 we are primarily concerned with the structure of L when $D(L) = I(L)$ ($I(L)$ denotes the inner derivations of L).

Two well-known facts which we shall use without further comment are 1) $D(L)$ is algebraic, and 2) If N is a nilpotent Lie algebra and U is a subspace of N such that $N = U + [N, N]$, $U \cap [N, N] = 0$, then U generates L.

Throughout, the ground field is assumed to have *characteristic* 0 and this implies that $t(L^*) = t(L)$ if L^* is obtained from L by extending the ground field. Dim (L) will denote the dimension of L over K, codim $(M) = \dim (L) - \dim (M)$ if M is a subspace of L.

2. **On** $t(L)$. Let L be a nilpotent Lie algebra with basis $x_1, \cdots, x_n$ and suppose $[x_i, x_j] = \sum_k c_{ijk}x_k$ with c_{ijk} in the ground field. We let $r\{x_1, \cdots, x_n\}$ denote the rank of the coefficient matrix of the linear equations $c_{ijk}(\lambda_i+\lambda_j-\lambda_k) = 0$ in the variables $\lambda_1, \cdots, \lambda_n$ as i, j, k run from 1 to n. Put $r\{L\}$ = minimum $r\{x_1, \cdots, x_n\}$ as $x_1, \cdots, x_n$ runs over all bases of L.

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