ALGEBRAS OF BOUNDED FUNCTIONS IN L_p

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Let (X, m) be a measure space as defined by Halmos in [4; 73], and let p be a positive number. We define $L_p(X, m)$ to be the class of all measurable real valued functions f on X for which $\int |f|^p \, dm < \infty$, two functions in $L_p(X, m)$ being identified if they are equal almost everywhere. Then $L_p(X, m)$ is a real topological vector space under the usual operations of addition and scalar multiplication of functions where a neighborhood system of the zero function θ is given by sets of the form $\{f \in L_p(X, m); \int |f|^p < \epsilon\}$ for all $\epsilon > 0$. Indeed $L_2(X, m)$ is a real Hilbert space with the inner product $(f, g) = \int fg$. More generally $L_p(X, m)$ is a real Banach space with norm $\| f \| = (\int |f|^p)^{1/p}$ if $p \geq 1$, but $L_p(X, m)$ might not be normed if $0 < p < 1$ (i.e., there might not exist a norm on the vector space $L_p(X, m)$ which induces the same topology on $L_p(X, m)$; see [2]).

The product of two bounded functions in $L_p(X, m)$ must be in $L_p(X, m)$. By an algebra of bounded functions in $L_p(X, m)$ we mean a collection of bounded functions in $L_p(X, m)$ closed under products and linear combinations. In the present paper we produce several density theorems concerning algebras of bounded functions in $L_p(X, m)$. Such density theorems are not new; in [3] Farrell showed that if X is locally compact, if m is a σ-finite Baire measure, if the algebra "separates" open sets in a certain sense and contains a function which is positive almost everywhere, then the algebra is dense in L_p. Theorem 3 in the present paper generalizes the work of Farrell.

The proof of the following Theorem is fairly routine if (X, m) is a finite or σ-finite measure space; however, this result is valid for an arbitrary measure space and any restrictions on (X, m) are immaterial. (Note that a sequence of measurable functions $\{f_n\}$ is said to "converge in measure" to a measurable function f on a set E if $\lim_n mE[|f - f_n| \geq \epsilon] = 0$ for every $\epsilon > 0$; see [6; 223].)

Theorem 1. *Let M be an algebra of bounded functions in $L_p(X, m)$ where (X, m) is a measure space, and let f be a function in $L_p(X, m)$. Then the following are equivalent.*

(1) *There is a sequence of functions $\{f_n\}$ in M such that* $\lim_n \int_X |f - f_n|^p = 0$.

(2) *For each measurable set E with $m(E) < \infty$, there is a sequence of functions $\{f_n\}$ in M for which* $\lim_n \int_E |f - f_n|^p = 0$.

(3) *For each measurable set E for which $m(E) < \infty$, there is a sequence of functions $\{f_n\}$ in M converging in measure to f on E.*

(4) *There is a sequence of functions $\{f_n\}$ in M converging in measure to f on X.*

In particular if f is a bounded function in $L_p(X, m)$ then these properties are equivalent to each of the following.