THE LEBESGUE DECOMPOSITION

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Our purpose is to present a Lebesgue decomposition theorem extending both C. E. Rickart's theorem: [2], and the theorem of [1]. This consists in establishing the theorem of [1] in a setting where the monotone norm condition of [1, Definition 2.3] is replaced by a form of C. E. Rickart's s-bounded condition [2, Lemma 2.2].

Suppose G is a generalized complete normed abelian group and T is a Boolean algebra of projection operators on G. (Throughout this paper we assume familiarity with [2] and [1].) Moreover, if each of a and b is an element of T, $a \leq b$, and $g \in G$, then we suppose $\| a(g) \| \leq \| b(g) \|$.

An element f of G is said to be *s-bounded* if for every sequence $\{t_i\}$ of pairwise disjoint elements of T, $t_i(f) \to 0$.

If a bounded element f of G has the property of Definition 2.3 of [1] with respect to T, then f is s-bounded. Moreover, the subspace of $l_\infty(I)$, where I denotes the set of positive integers, consisting of the sequences converging to zero, together with the projections on finite subsets of I and their complements (see Example 2.1 of [1]), comprise an example to illustrate that s-boundedness is, in fact, a weaker condition on bounded elements than we used in [1].

C. E. Rickart's theorem fits into our setting as follows: The elements g of G are vector-valued functions defined on a "nice system" S of sets. The elements t of T are projections induced by the elements E of S (i.e., if t corresponds to E, then for every $F \in S$, $(tg)(F) = g(E \cap F)$). The norm $||| g |||$ of the elements g of G is $\sup \{\| g(E) \|; E \in S\}$. Finally,

$$||| t'(g) ||| = \sup \{\| g(F - E) \|; F \in S\},$$

where t corresponds to E and t' denotes the complement of t.

Henceforth f shall denote a bounded and s-bounded element of G.

Lemma 1. *If $\{t_i\}$ is a monotone sequence of elements of T, then $\lim_i t_i(f)$ exists.*

Proof. Suppose, for instance, $\{t_i\} \uparrow$ in T. Moreover, suppose $\lim_i t_i(f)$ does not exist. Then there exists $\epsilon > 0$ and an increasing sequence $\{n_k\}$ of positive integers such that $\| (t_{n_{k+1}} - t_{n_k})(f) \| > \epsilon$. But this contradicts our basic supposition that f be s-bounded.

Lemma 2. *If $\{t_k\}$ is a sequence of elements of T and $\epsilon > 0$, then there exists a positive integer n such that if $j \geq i > n$, then*

$$\| (\bigvee_{i \leq k \leq j} t_k - \bigvee_{k \leq n} t_k)(f) \| < \epsilon.$$

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