THE ZEROS OF THE PARTIAL SUMS OF CERTAIN SMALL ENTIRE FUNCTIONS

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1. **Introduction, results and conjectures.** Let $\sum a_n z^n$ be a formal power series, $s_n(z)$ its partial sum of order n. It is known from the work of Carlson [2], Rosenbloom [7] and Korevaar [4] that if every s_n is free of zeros in some angle with vertex O and fixed opening, then the power series represents an entire function of order 0. Actually a little more is known: McCoy [5] has recently shown that

$$a_n = O\{\exp(-cn \log n \log \log n)\}$$

for some $c > 0$ depending on the size of the angle. However, in view of Edrei's result [3] that $a_n = O\{\exp(-cn^2)\}$ in the case of a zero free half plane, it is reasonable to conjecture that also in the general case the a_n are only very little larger than

$$O\{\exp(-cn^2)\}.$$

No examples seem to be known which shed additional light on the situation. The present paper fills this gap, and leads to a more detailed conjecture. (Added July 1962: Ganelius has just proved these conjectures (see the paper following this one).)

Let

$$f(z) = \sum_0^\infty e^{-n^\alpha} z^n, \quad \text{where} \quad \alpha > 1. \tag{1.1}$$

From here on s_n will denote the partial sums of this series:

$$s_n(z) = \sum_0^n e^{-k^\alpha} z^k. \tag{1.2}$$

When $\alpha \geq 2$, all zeros of all the s_n are negative real [6; 69, problem 176].

In this paper *we take* $1 < \alpha < 2$ and we obtain detailed results on the growth and the zeros of f and the partial sums s_n.

An estimate for f on the negative real axis provides a lower bound for the number of negative real zeros of f of absolute value $\leq r$. An estimate for f on circles about O and Jensen's formula provide an upper bound for the total number of zeros of absolute value $\leq r$. Combining these results we find that *all but a finite number of the zeros of f are negative real.* The necessary estimates for f are obtained with the aid of Poisson's transformation.

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